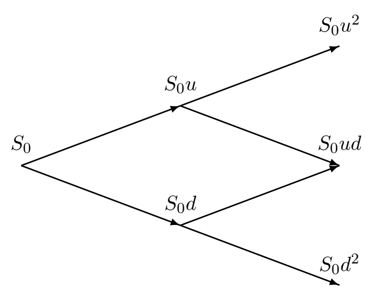
Stochastic Processes: The Fundamentals

Exam

October 21, 2020

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 60% of your final grade.

Question 1: Binomial trees (20 points)
We want to price a European put-option on the S&P-500 index. Strike price of the option is USD 3,300 and the remaining time to maturity is one year. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a **2-step binomial tree** where the two steps have equal length of six months. The probability of an upward movement is p. The current value of the S&P-500 index is USD 3,500. The risk free rate r is 1.5% per annum with continuous compounding.



(a) Suppose that $p = \frac{5}{8}$ and $u = 1.125$. Determine d and provide the resulting tree, where the following conditions should be satisfied (**5 pts**):

- $d < (1 + r) < u$
- the expected 1-year **simple net return** on the S&P-500 index is 5%.

Note: if $S_0 = 3{,}500$ and $S_1 = 3{,}850$, then the **simple net return** equals 3,850/3,500 - 1 = 10%.

(b) Using the binomial tree, calculate the no-arbitrage price of a European put-option on the S&P-500 index with strike price USD 3,300 and a remaining time to maturity of 1 year. If you were not able to solve (a) then choose $d = 1/u$. (**10 pts**)

(c) Calculate delta for all relevant nodes in the tree and use these deltas to show that the no-arbitrage value calculated in (b) is indeed the amount of money that is necessary to replicate the payoff of the option at maturity. To be more specific: start with a cash value equal to USD 0 and show for the path that leads to $S_0 d^2$ that the value of the replicating portfolio at the maturity date of the option is lower than the payoff of the option and relate the difference to the option price (**5 pts**).

Time indication: 30 mins

Question 2: Equity forwards (10 pts)
Consider an equity forward contract on the S&P-500 index. Suppose that the current value of the S&P-500 index is 3,500. Assume that the index pays no dividends and that the risk free rate is 1.5% per annum with continuous compounding. An institutional investor contacts an investment bank and requests for a trade in which the institutional investor buys the S&P-500 index for 3,500 in 6 months from the current date. The investment bank agrees to enter into this deal. What is the no-arbitrage value of this deal, i.e., what is the amount of money the investment bank wants to receive from the institutional investor? You can assume frictionless markets.

Time indication: 10 mins

Question 3: Itô's Lemma (15 pts)
Consider stochastic process $(X_t)_{t\geq 0}$ with dynamics:

$$dX_t = (t+1)X_t dt + \frac{1}{2}X_t dW_t,$$

where W is a standard Brownian Motion process.

(a) Derive the stochastic differential equation for the process of $\log X$. (**5 pts**)

Consider now:

$$Y_t = X_t e^{-\frac{1}{2}(t+1)^2}.$$

(b) Determine the stochastic differential equation of Y_t (**5 pts**).

(c) Is the stochastic process $(Y_t)_{t\geq 0}$ a martingale? Please motivate your answer (**5 pts**).

Time indication: 10 mins

Question 4: Itô isometry (10 pts)
Let W be a standard Brownian Motion process. Use Itô isometry to compute (**10 pts**):

$$\mathbb{E}\left[\left(\int_0^t (1+W_s)dW_s\right)^2\right].$$

Time indication: 10 mins

Question 5: Pricing kernel (10 pts)
Consider a standard Black-Scholes world with $\mu > r$, i.e. a positive risk premium on stock price risk. In this world, the pricing kernel process is given by the following stochastic differential equation:

$$d\pi_t = -r\pi_t dt - \left(\frac{\mu - r}{\sigma}\right)\pi_t dW_t, \quad \pi_0 = 1,$$

where W is a Brownian Motion under the real-world probability measure.

Show that the solution of this SDE is given by:

$$\pi_t = \pi_0 e^{-rt-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t-\left(\frac{\mu-r}{\sigma}\right)W_t}.$$

Time indication: 10 mins

Question 6: Payoff pricing (15 pts)
Consider a standard Black-Scholes world. At time t we want to find the no-arbitrage price of the following payoff structure at time $T > t$:

$$V(T) = \begin{cases} 0 & \text{if } S(T) < K, \\ \dfrac{K}{S(T)} & \text{if } S(T) \geq K, \end{cases}$$

where $K > 0$ is a constant. Derive the no-arbitrage price of the contract, $V(t)$, in the Black-Scholes world.

Time indication: 30 mins

Question 7: Black-Scholes pricing (10 pts)
Again, consider a standard Black-Scholes world with $\mu > r$, i.e. a positive risk premium on stock price risk. We know from the lecture slides that prices of European call-options and European put-options do not depend on the growth rate μ of the underlying stock S. This seems weird because a call-option, for instance, provides positive exposure to the underlying stock, i.e. if the underlying stock moves up then the value of the call-option moves up as well. Given this positive exposure to stock price movements, one might think that the price of a call-option should be related to the stock's growth rate μ. Explain very carefully why a higher μ does not lead to a change in the no-arbitrage price of a call-option in the Black-Scholes model.

Time indication: 5 mins