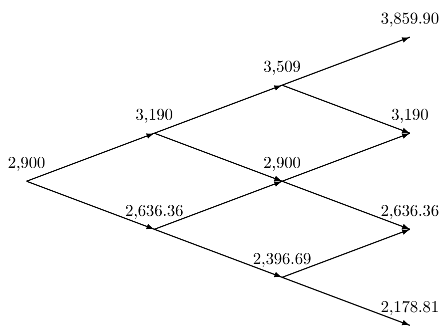
Stochastic Processes: The Fundamentals

Exam

December 13, 2019

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 70% of your final grade.

Question 1: Binomial trees (22 points)
We want to price a European and American put-option on the S&P-500 index. Strike price of both options is USD 2,900 and the remaining time to maturity is 3 months. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a 3-step binomial tree. The tree is given in the following picture:



In this tree I have used $u = 1.10$ and $d = 1/u$.

(a) Assume that the real-world probability of an upward movement is $\frac{4}{7}$. Calculate the variance of the 3-months return on the S&P-500 index (**5 pts**).

(b) Is the stochastic process of the S&P-500 index a Markov process? Please motivate your answer (**4 pts**).

(c) Calculate the no-arbitrage value of European and American put-option using the risk-neutral valuation method. As was already mentioned, the strike price of the option is USD 2,900 and the remaining time to maturity is three months. Assume that the risk-free interest rate per 1 month is 0%, i.e. an investment of USD 1,000 in a money market account that pays this interest rate delivers USD 0 interest after one month (**7 pts**).

Suppose that the market price of the European put-option is USD 250.

(d) Provide for the path that leads to the lowest value for the S&P-500 index after 3 months, the arbitrage strategy in detail. I.e. give the exact strategy at times $t = 0, 1, 2$ and show numerically that the payoff of the strategy is positive in this particular path. (**6 pts**).

Question 2: Algebras (10 points)
Let a, b, c be three distinct points.

(a) Write down all algebras on $\Omega = \{a, b\}$ (**3 pts**).

(b) Write down all algebras on $\Omega = \{a, b, c\}$ (**4 pts**).

(c) Give an explicit counterexample which shows that the union of algebras is not necessarily an algebra (**3 pts**).

Question 3: Brownian Motion (18 points)
Let $W(t)$ be a Brownian Motion and define

$$B(t) = \int_0^t \text{sign}(W(s))dW(s),$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

(a) Compute $d[B(t)W(t)]$ and conclude that $\mathbb{E}_0[B(t)W(t)]$ is equal to zero. *Hint: write* $d[B(t)W(t)]$ *in integral form and then take expectations.* (**6 pts**)

(b) Verify that: (**5 pts**)

$$dW^2(t) = 2W(t)dW(t) + dt.$$

(c) Compute $d[B(t)W^2(t)]$ and use the result to compute $\mathbb{E}_0[B(t)W^2(t)]$ (**7 pts**).

Question 4: Stochastic differential equations (19 points)
Suppose we have the standard Black-Scholes market that consists of a stock S and a money market account B. The stochastic differential equations of this market are given by:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^S, \quad S_0 = s \\ dB_t &= r B_t dt, \quad B_0 = 1, \end{aligned}$$

where W^S is a Brownian Motion under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

(a) Show that the solution of the SDE for the stock price S is given by (**6 pts**):

$$S_t = s e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Let us now introduce stochastic volatility into the model:

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S, \quad S_0 = s \\
d\sigma_t^2 &= -\kappa(\sigma_t^2 - \sigma^2)dt + \gamma dW_t^V, \quad \sigma_0^2 = \sigma^2 \\
d[W^S, W^V]_t &= \rho dt \\
dB_t &= rB_t dt, \quad B_0 = 1,
\end{aligned}$$

where W^S and W^V are Brownian Motions under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

It is possible to derive that:

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2 \int_0^t e^{\kappa u} du + \gamma \int_0^t e^{\kappa u} dW_u^V\right\}.$$

(b) Derive $\mathrm{Var}_0(\sigma_t^2)$ (**6 pts**).

Now we are interested in the process of the squared variance.

(c) Define $Y_t = \sigma_t^2$. Derive dY_t^2. Is this process also mean-reverting? (**7 pts**)

Question 5: Bachelier process (21 points)
Consider the following market:

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma dW_t, \quad S_0 = s \\
dB_t &= rB_t dt, \quad B_0 = 1, \quad r > 0,
\end{aligned}$$

where W is a Brownian Motion under the real-world probability measure $\mathbb{P}$, S denotes the stock price process and B the money market account.

The solution of the stock price SDE is given by:

$$S_t = e^{\mu t} S_0 + \sigma e^{\mu t} \int_0^t e^{-\mu s} dW_s.$$

(a) Prove that this is indeed the solution of the stock price SDE (**8 pts**). *Hint: write the solution as*

$$X_t = Y_t + Z_t \cdot R_t,$$

where,

$$\begin{aligned}
Y_t &= e^{\mu t} \cdot S_0 \\
Z_t &= e^{\mu t} \cdot \sigma \\
R_t &= \int_0^t e^{-\mu s} dW_s.
\end{aligned}$$

First compute the differentials dZ, dY and dR. Then use the multidimensional Itô formula to the function $f(y, z, r) = y + z \cdot r$.

It is possible to derive that:

$$d\left(\frac{S_t}{B_t}\right) = (\mu - r)\left(\frac{S_t}{B_t}\right)dt + \sigma\frac{1}{B_t}dW_t.$$

We can rewrite this SDE as:

$$d\left(\frac{S_t}{B_t}\right) = \frac{\sigma}{B_t}\left(dW_t + \frac{(\mu - r)S_t}{\sigma}dt\right).$$

Using Girsanov's theorem we can derive the process for the discounted stock price under the risk-neutral probability measure.

$$d\left(\frac{S_t}{B_t}\right) = \frac{\sigma}{B_t}d\tilde{W}_t.$$

(b) Derive the SDE for the stock price under the risk-neutral measure by calculating $d(Y_t \cdot B_t)$, where $Y_t := \left(\frac{S_t}{B_t}\right)$. Is the resulting process for S a martingale? (**5 pts**).

The probability distribution of the stock price at time T conditional on time t information is given by:

$$S_T|S_t \sim N\left(e^{r(T-t)}S_0, \frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)\right).$$

(c) Use this to calculate the no-arbitrage price at time t of a European put option with strike price K and maturity date $T > t$ (**8 pts**).