vrije Universiteit amsterdam

School of Business and Economics

Exam:	Stochastic Processes: The Fundamentals
Code:	E_FIN_SPFUN
Examinator:	dr. M.J. Boes
Co-reader:	dr. A. Opschoor
Date:	October 23, 2019
Time:	15:15
Duration:	2 hours and 45 minutes
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	20
Type of questions:	Open
Answer in:	English

Remarks: Students are allowed to bring a cheat-sheet. This sheet has maximum size of A4 and is written only on one side.

Students are allowed to take the exam home.

Credit score:	90 credits counts for a 10
Grades:	The grades will be made public within 10 working days.
Inspection:	To be announced on Canvas.
Number of pages:	4

Good luck!

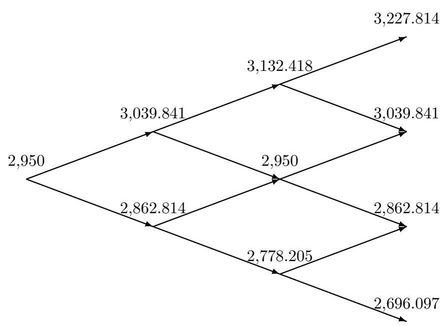
Stochastic Processes: The Fundamentals

Exam

October 23, 2019

Grading: you can earn 90 points with this exam. Your exam grade is then: $\frac{\#\text{points}}{10} + 1$. The exam counts for 70% of your final grade.

Question 1: Binomial trees (22 points)
We want to price a European call-option on the S&P-500 index. Strike price of the option is USD 3,000 and the remaining time to maturity is six weeks. Assume that the index pays no dividends. We model the evolution of the S&P-500 index by means of a 3-step binomial tree. The tree is given in the following picture:



In this tree I have used $u = 1.030455$ and $d = 1/u$.

(a) Assume that the real-world probability of an upward movement is $\frac{11}{20}$. Calculate the expectation of the 6-weeks return on the S&P-500 index (**4 pts**).

(b) Is the stochastic process of the S&P-500 index a martingale? Please motivate your answer (**4 pts**).

Denote the index values at time $t = 0$, $t = 2$ weeks, $t = 4$ weeks and $t = 6$ weeks by S_0, S_1, S_2, and S_3, respectively.

(c) Calculate $\mathbb{E}(S_3|S_1)$ (**3 pts**).

(d) Calculate the no-arbitrage value of the European call-option using the risk-neutral valuation method. As was already mentioned, the strike price of the option is USD 3,000 and the remaining time to maturity is six weeks. Assume that the risk-free interest rate per two weeks is 0.10%, i.e. an investment of USD 1,000 in a money market account that pays this interest rate delivers USD 1 interest after two weeks (**6 pts**).

(e) Suppose the market price of the call-option is higher than the price you calculated in (d). This implies the existence of an arbitrage strategy. Give this strategy and show that the strategy generates a riskless profit (**5 pts**).

Question 2: Algebras (10 points)
Let $X := \{a, b, c\}$, $A_1 := \{\emptyset, X, \{a\}, \{b, c\}\}$ and $A_2 := \{\emptyset, X, \{b\}, \{a, c\}\}$.

(a) Is A_1 an algebra? Please motivate your answer (**3 pts**).

(b) Is A_2 an algebra? Please motivate your answer (**3 pts**).

(c) Is $A_1 \cup A_2$ an algebra? Please motivate your answer (**4 pts**).

Question 3: Brownian Motion (16 points)
Define the stochastic process Z by $Z(t) = W^4(t)$, $t \geq 0$, where $W(t)$, $t \geq 0$ is a Brownian Motion.

(a) Derive the stochastic differential equation of Z. What is the initial value? (**6 pts**)

(b) Compute $\mathbb{E}_0[Z(t)]$ and conclude that Z is not a martingale (**5 pts**).

(c) Provide a stochastic process Y that consists of a random part which is given by Z and a deterministic part. Choose the deterministic part such that Y is a martingale (**5 pts**).

Question 4: Stochastic differential equations (20 points)
Suppose we have a market that consists of a stock S and a money market account B. The stochastic differential equations of this market are given by:

$$\begin{aligned}
dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S, \quad S_0 = s \\
d\sigma_t^2 &= -\kappa(\sigma_t^2 - \sigma^2)dt + \sigma^\sigma \sigma_t dW_t^V, \quad \sigma_0^2 = \sigma^2 \\
d[W^S, W^V]_t &= \rho dt \\
dB_t &= rB_t dt, \quad B_0 = 1,
\end{aligned}$$

where W^S and W^V are Brownian Motions under the real-world probability measure $\mathbb{P}$ and r the continuously compounded risk free interest rate.

(a) Is this market complete? Please motivate your answer. How could this market be completed? (**4 pts**)

Let us now focus on the SDE for the variance.

(b) Derive $d\exp(\kappa t)\sigma_t^2$ and show that (**6 pts**):

$$\sigma_t^2 = e^{-\kappa t}\left\{\sigma_0^2 + \kappa\sigma^2 \int_0^t e^{\kappa u} du + \int_0^t e^{\kappa u}\sigma^\sigma \sigma_u dW_u^V\right\}.$$

(c) Derive the expression for σ_{t+h}^2 conditional on σ_t^2 (**5 pts**).

(d) Compute $\mathbb{E}_t[\sigma^2_{t+h}]$ (**5 pts**).

Question 5: Bachelier process (22 points)
Consider the following market:

$$dS_t = \mu S_t dt + \sigma dW_t, \quad S_0 = s$$
$$dB_t = rB_t dt, \quad B_0 = 1, \quad r > 0,$$

where W is a Brownian Motion under the real-world probability measure $\mathbb{P}$, S denotes the stock price process and B the money market account.

Suppose now that the solution of the stock price SDE is given by:

$$S_t = e^{\mu t} S_0 + \sigma e^{\mu t} \int_0^t e^{-\mu s} dW_s.$$

(a) Give the probability distribution of $S_t | S_0$ (**5 pts**).

(b) Give your personal opinion on this model for stock price behaviour (**3 pts**).

(c) Use Itô's lemma to derive the process for the discounted stock price $Y_t := S_t / B_t$ (**3 pts**).

(d) Apply Girsanov's theorem and conclude that the stock price process for S under the risk-neutral probability measure is given by:

$$dS_t = rS_t dt + \sigma d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian Motion under the risk-neutral probability measure $\mathbb{Q}$ (**4 pts**).

In the same spirit as above, the solution of this SDE is given by:

$$S_t = e^{rt} S_0 + \sigma e^{rt} \int_0^t e^{-rs} d\tilde{W}_s.$$

(e) Calculate the no-arbitrage price at time t of a European digital call option with strike price K and maturity date $T > t$. This option pays 1 if the stock price at maturity is larger than strike price K and 0 in all other cases (**7 pts**).