Solutions to the Trial Exam SPF II

Question 1

Suppose that the risk free rate is 2% per year and the average dividend yield on the AEX stock index is 5% per year.

a) The current index value is 494 Euros and the 6-months futures contract on this index is quoted at 503 Euros. What arbitrage opportunity does this create and how can you realize it?
b) A UK investor would like to enter a pound sterling-valued forward contract on this index, expiring in 6 months. Current exchange rate is 1.3, suppose again that interest rates in EU are 2% and interest rates in UK are 2.5%. What is the forward price of such a contract and how the seller would hedge this contract?

Solution:

a) Forward price should be *F(0,6)=494*exp[(0.02-0.05)*1/2]=486.6*, so the forward contract is very overpriced. The arbitrage opportunity can be realized by shorting the forward contract and buying the index and waiting 6 months until expiry. The risk free profit is *503-484.6=16.34.*

b) The euro forward price is 486.6 euros, so that is the amount the Sterling investor will need in 6 months. Now this amount is equivalent to *486.6*exp(-0.02/2)=481.73* euros. So Sterling investor buys this amount now for *481.73/1.3=370.56* pounds and holds those euros on EU bank account. So the fair forward price is then *370.56*exp(0.025/2)=375.22* pounds.

Question 2.

Prove the following bound for European call options on a stock without dividend, using portfolio argument:

$$C_2 \leq \frac{E_3 - E_2}{E_3 - E_1} C_1 + \frac{E_2 - E_1}{E_3 - E_1} C_3,$$

where C_1, C_2 and C_3 are calls with the same expiry, T, and have exercise prices E_1, E_2 and E_3 respectively, where $E_1 < E_2 < E_3$.

Hint: Consider $E_2 = \lambda E_1 + (1 - \lambda) E_3$.

Solution:

$\Pi = -C_2 + \lambda C_1 + (1-\lambda)C_3$:
(where we have chosen λ such that $E_2 = \lambda E_1 + (1-\lambda)E_3$).
Then

$$\lambda = \frac{E_3 - E_2}{E_3 - E_1} \text{ and } (1-\lambda) = \frac{E_2 - E_1}{E_3 - E_1}.$$

At time T, the portfolio is worth

$$\Pi(T) = -\max(S - E_2, 0) + \lambda \max(S - E_1, 0) + (1-\lambda)\max(S - E_3, 0).$$

On inspection, we see that

$$\Pi(T) \geq 0.$$

In the absence of arbitrage opportunities, we must therefore have

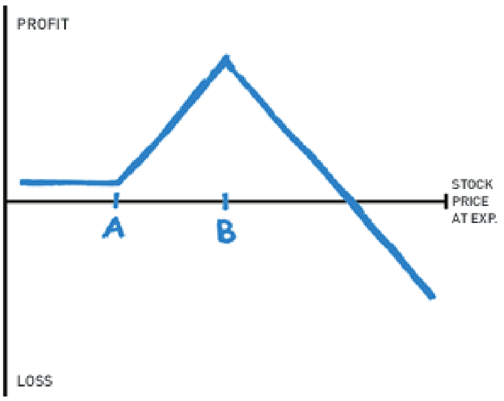
$$\Pi(t) = -C_2 + \lambda C_1 + (1-\lambda)C_3 \geq 0.$$

Substituting for λ, we find

$$C_2 \leq \frac{E_3 - E_2}{E_3 - E_1} C_1 + \frac{E_2 - E_1}{E_3 - E_1} C_3.$$

Question 3.

a) Consider the payoff of the following options package:



Explain how to construct this payoff using regular calls. Explain what is your "bet" about the underlying stock price, if you purchase such a package, and what is a likely current value of the stock price when this package is first purchased, in terms of A and B. What event will put you in a load of trouble, if you own such a package?

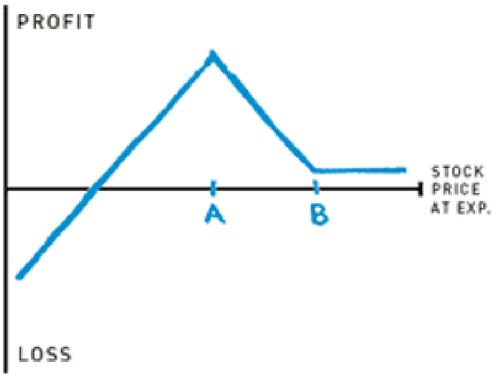
Solution:

- Buy a call, strike price A
- Sell two calls, strike price B

Generally, the stock will be below or at strike A

You're slightly bullish. You want the stock to rise to strike B and then stop. Ideally, you want a slight rise in stock price to strike B. Although one of the calls you sold is "covered" by the call you buy with strike A, the second call you sold is "uncovered," exposing you to theoretically unlimited risk. If the stock price goes too high, you'll experience large loss.

b) Now consider the payoff of a similar package:



Explain how to construct this payoff using regular puts. Explain what is your "bet" about the underlying stock price, if you purchase such a package and what is a likely current value of the stock price when this package is first purchased, in terms of A and B. What event will put you in a load of trouble, if you own such a package?

Solution:

- Sell two puts, strike price A
- Buy a put, strike price B

Generally, the stock will be at or above strike B.

You're slightly bearish. You want the stock to go down to strike A and then stop. Ideally, you want a slight dip in stock price to strike A. Although one of the puts you sold is "covered" by the put you buy with strike B, the second put you sold is "uncovered," exposing you to significant downside risk. If the stock goes too low, you'll experience a large loss.

Question 4.

Consider the following modification of a **binary option:** is a claim which pays 1 Euro if the stock price at maturity falls within some pre-specified interval: $\alpha<S(T)<\beta$. Otherwise nothing will be paid out. Determine the arbitrage free price of such a binary option.
Hint: use the approach and the prices of usual binary options.

Solution:

Note that the payoff of this option is given by:

$$\Phi_B(S_T; \alpha, \beta) = \mathbb{I}(S_T > \alpha) - \mathbb{I}(S_T > \beta).$$

This is of course simply the payoff of the package consisting of long position in binary call with strike α and a short position in binary call with strike β. From lectures and exercises we know what is the price of a binary option with strike α, it is

$$e^{-rT}\mathcal{N}(d_2(\alpha)).$$

So the price of the package and hence of the option in question is:

$$V_B(0; \alpha, \beta) = e^{-rT}\left[\mathcal{N}\left(d_2(\alpha)\right) - \mathcal{N}\left(d_2(\beta)\right)\right]$$

Question 5. Perpetual American option.

Solution:

(a)

$$\begin{aligned} dP &= P_0\lambda S^{\lambda-1}dS + \frac{1}{2}P_0\lambda(\lambda-1)S^{\lambda-2}dS^2 \\ &= (\lambda\mu + \frac{1}{2}\lambda(\lambda-1)\sigma^2)Pdt + \lambda\sigma PdW \\ &= \mu_P Pdt + \sigma_P PdW \end{aligned}$$

(b) The Black–Scholes equation is

$$\begin{aligned} rSP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} &= rP \\ P(t,S)|_{S=L} &= K - L \end{aligned}$$

$$\begin{aligned} rS\lambda S^{\lambda-1} + \frac{1}{2}\sigma^2 S^2\lambda(\lambda-1)S^{\lambda-2} &= rS^\lambda \\ P_0 L^\lambda &= K - L \end{aligned}$$

$$\begin{aligned} \lambda &= -\frac{2r}{\sigma^2} \\ P_0 &= \frac{K-L}{L^\lambda} \end{aligned}$$

Question 6. Geometric basket option

Solution (mixed up letters, sorry: c=d and d=c):

(a)

$$\begin{aligned} dY &= \frac{\partial Y}{\partial S_1} dS_1 + \frac{\partial Y}{\partial S_2} dS_2 + \frac{1}{2}\frac{\partial^2 Y}{\partial^2 S_1} dS_1^2 + \frac{1}{2}\frac{\partial^2 Y}{\partial^2 S_2} dS_2^2 \\ &= \frac{1}{2}\sqrt{\frac{S_2}{S_1}} dS_1 + \frac{1}{2}\sqrt{\frac{S_1}{S_2}} dS_2 - \frac{1}{8}\sqrt{\frac{S_2}{S_1^3}} dS_1^2 - \frac{1}{8}\sqrt{\frac{S_1}{S_2^3}} dS_2^2 \\ &= \frac{1}{2} Y \left(r dt + \sigma_1 dW_1 + r dt + \sigma_2 dW_2 \right) - \frac{1}{8} Y \left(\sigma_1^2 + \sigma_2^2 \right) dt \\ &= \left(r - \frac{1}{2}\sigma_3^2 \right) Y dt + \sigma_3 Y dW_3 \end{aligned}$$

where

$$\begin{aligned} \sigma_1 W_1 + \sigma_2 W_2 &= \sqrt{\sigma_1^2 + \sigma_2^2}\, W_3 \\ \sigma_3 &= \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

(b)

$$\begin{aligned} Y(T) &= \sqrt{s_1 s_2} \exp\left\{ \left(r - \frac{1}{2}\sigma_3^2 - \frac{1}{2}\sigma_3^2 \right) T + \sigma_3 W_3(T) \right\} \\ &= \sqrt{s_1 s_2} \exp\left\{ \left(r - \sigma_3^2 \right) T + \sigma_3 W_3(T) \right\} \\ \sqrt{S_1(T) S_2(T)} &= \sqrt{s_1 s_2} \exp\left\{ \frac{1}{2}\left(rT - \frac{1}{2}\sigma_1^2 T + \sigma_1 W_1(T) + rT - \frac{1}{2}\sigma_2^2 T + \sigma_2 W_2(T) \right) \right\} \\ &= \sqrt{s_1 s_2} \exp\left\{ rT - \frac{1}{4}(\sigma_1^2 + \sigma_2^2) T + \frac{1}{2}(\sigma_1 W_1 + \sigma_2 W_2) \right\} \\ &= \sqrt{s_1 s_2} \exp\left\{ \left(r - \sigma_3^2 \right) T + \sigma_3 W_3(T) \right\} = Y(T) \end{aligned}$$

(c)

$$\begin{aligned} V_F(0,T) &= e^{-rT}\mathbb{E}_0\left[Y(T) - F \right] = e^{-rT}\mathbb{E}_0[Y(T)] - e^{-rT} F \\ &= e^{-rT}\mathbb{E}_0\left[\sqrt{s_1 s_2} \exp\left\{ \left(r - \sigma_3^2 \right) T + \sigma_3 W_3(T) \right\} \right] - e^{-rT} F \\ &= e^{-rT}\sqrt{s_1 s_2} e^{(r - \frac{1}{2}\sigma_3^2)T} \mathbb{E}_0\left[\exp\left\{ -\frac{1}{2}\sigma_3^2 T + \sigma_3 W_3(T) \right\} \right] - e^{-rT} F \\ &= \sqrt{s_1 s_2} e^{-\frac{1}{2}\sigma_3^2 T} - e^{-rT} F \end{aligned}$$

(d)

$$\begin{aligned} \tilde{S} &= \sqrt{s_1 s_2} \\ \tilde{r} &= r \\ \tilde{\delta} &= \frac{1}{2}\sigma_3^2 \\ \tilde{\sigma} &= \sigma_3 = \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2} \end{aligned}$$