Exam "Stochastic Processes for Finance"

December 20, 2013

Give crisp and clear computations/argumentations.
Use of notes, textbook(s), calculators (and other electronic equipment) is not allowed.

This exam consists of five exercises of equal weight.
The final grade of this course is based on the average grade for the homework assignments (twenty percent) and the grade for this exam (eighty percent).

1. Consider the N-period binomial model, where at each step the stock price is multiplied by $u = 2$ if the outcome of the coin toss is H (with happens with probability $p \in (0, 1)$), and multiplied by $d = 1/2$, if the outcome is T. Now suppose we have a derivative security which pays, at time N, the amount K if the number of coin tosses with outcome H is even, and the amount 0 otherwise. Let the interest rate be $r = 1/4$.

(a) Compute the price of the derivative security at time j, for $0 \leq j \leq N-1$.

(b) In the replicating portfolio, how much stock is held at times $0, 1, \cdots, N-2$? (In other words, compute Δ_j for $0 \leq j \leq N-2$).

2. Let $T > 0$ and consider the following boundary value problem:

$$\frac{\partial f(t,x)}{\partial t} + 2t\frac{\partial f(t,x)}{\partial x} + \frac{1}{2}t^4\frac{\partial^2 f(t,x)}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t \in [0,T],$$

$$f(T,x) = (x+1)^2, \quad x \in \mathbb{R}.$$

Use the Feynman-Kac theorem to find an explicit solution, and check this solution.

3. (a) For which value or values (if any at all) of a is the process

$$W^3(t) - a\int_0^t W(s)\,ds, \quad t \geq 0,$$

a martingale?

(b) Is the process $W^4(t) - 3t^2,\ t \geq 0$, a martingale?

4. Consider the standard Black-Scholes model for a stock, with parameters α and σ. As usual, $S(t)$ denotes the stock price at time t. Suppose we have a derivative security which pays, at the time of maturity T, the amount K if the maximum stock price over the interval $[0, T]$ is between $2S(0)$ and $4S(0)$, and the amount 0 otherwise.
Compute the price at time 0 of this derivative security, for the case where $\sigma = 1$ and the interest rate r is equal to $1/2$.
Remark: You may express the result in terms of the function $N(x)$, the probability that a random variable with standard normal distribution has value smaller than x.

5. Consider again the standard Black-Scholes model for a certain stock. Let $T > 0$. Let, for $x \geq 0$, $c(x)$ denote the price at time 0 of a European call option with strike price x and time of maturity T. Now let $K \geq 0$ and consider a derivative security which pays, at time T, the amount $\Phi(S(T))$, where

$$\Phi(y) = \begin{cases} 0, & y < K \\ y - K, & y \in [K, 2K) \\ K, & y \geq 2K \end{cases}$$

Express the price at time 0 of this derivative security in terms of the function c defined above.