WRITTEN EXAM: STOCHASTIC PROCESSES FOR FINANCE (CODE X_400352)

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General information. The written exam consists of 7 problems with a varying number of questions in each of them. Provide clear, but brief answers to the questions. Use of books or notes is not allowed.

Grading. The final grade for the course is based on the average grade for homework assignments (50%) and the grade for the written exam (50%). In greater detail, it is computed as follows: each problem of the written exam has an initial weight of 10 points. The number of points for all problems (the maximum is 70) is summed up and multiplied by a factor 1/7 so as to normalise the grade for the written exam to the interval $[0, 10]$. Denoting by x the normalised grade for the written exam and by y the average grade for the homework assignments (also normalised to the interval $[0, 10]$), an average of the two is computed as

$$z = 0.5 \times x + 0.5 \times y.$$

The final grade is obtained by rounding z off according to the faculty regulations.

Good luck. This seems to be never superfluous. Good luck!

Problem 1. Provide answers to the following two questions.

(i) Let a stochastic process $X = (X(t))_{t \geq 0}$ be adapted to a filtration (flow of information) $\{\mathcal{F}_t\}_{t \geq 0}$. When is X an $\mathcal{F}_t$-martingale?

(ii) Consider a stochastic process

$$X(t) = e^{3.14W(t)+g(t)},$$

where W is a Wiener process, while g is a deterministic function. The process X is adapted to the filtration $\{\mathcal{F}_t^W\}_{t \geq 0}$ generated by the Wiener process W (you do not need to prove this fact). It is obvious that for a general function g the process X is not a $\mathcal{F}_t^W$-martingale. Find a function g, such that the process X is a martingale.

Problem 2. Consider a stochastic differential equation

$$dX(t) = X(t)dt + dW(t),$$
$$X(0) = 0,$$

and let a stochastic process $\widetilde{X}$ be defined by

$$\widetilde{X}(t) = e^t \int_0^t e^{-s} dW(s).$$

Provide answers to the following two questions.

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(i) Show that the process $\widetilde{X}$ satisfies the stochastic differential equation above. Hint: write $\widetilde{X}(t) = Z(t)R(t)$ for

$$Z(t) = e^t, \quad R(t) = \int_0^t e^{-s} dW(s)$$

and next apply a multidimensional version of Itô's formula (with $f(z,r) = zr$) to $\widetilde{X}$.

(ii) Compute the expectation $\mathbb{E}[\widetilde{X}(t)]$ and the variance $\mathbb{V}\mathrm{ar}[\widetilde{X}(t)]$ of $\widetilde{X}(t)$.

Problem 3. Consider the boundary value problem

$$\frac{\partial F(t,x)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F(t,x)}{\partial x^2} = 0, \quad x \in \mathbb{R}, t \in [0,T),$$
$$F(T,x) = x^3.$$

Provide answers to the following two questions.

(i) Using the Feynman-Kac formula, find a solution to the above boundary value problem.

(ii) Check by a direct computation that a solution found through the Feynman-Kac formula indeed solves the above boundary value problem.

Problem 4. Provide answers to the following three questions.

(i) Describe the Black-Scholes model of the stock market.

(ii) In the Black-Scholes model setting let $\mathcal{X}$ be a contingent claim, a fair (i.e. Black-Scholes) price of which at time t is given by $F(t, S(t))$ with $S(t)$ denoting the stock price at time t. The pricing function corresponding to $\mathcal{X}$ is thus $F(t,s)$. Give the definition of the Greeks $\Delta, \Gamma, \rho, \Theta$ and $\mathcal{V}$ corresponding to this pricing function F.

(iii) Show that the Greeks corresponding to the pricing function F above satisfy the relationship

$$\Theta + \frac{1}{2}\sigma^2 s^2 \Gamma = rF - rs\Delta.$$

Problem 5. Suppose a Black-Scholes model with parameters $\alpha = 5$, $\sigma = 1$ and $r = 0$ is given, where α is the local mean rate of return of the stock, σ is its volatility and r is the interest rate. Consider a simple claim $\mathcal{X}$ that at time T pays 1 dollar if the stock price $S(T)$ at time T is above a predetermined (at the conclusion of the contract at time $t = 0$) level $K > 0$ and pays zero otherwise. The payoff function Φ of this option is thus given by

$$\Phi(s) = 1_{[s>K]} = \begin{cases} 1 & \text{if } s > K, \\ 0 & \text{if } s \leq K. \end{cases}$$

Provide an answer to the following question.

(i) Compute explicitly a fair (Black-Scholes) price $\Pi(0)$ at time $t = 0$ of the claim $\mathcal{X}$ described above. Hint: it is helpful to know that for a random variable Y and a constant y, assuming the expectation is computed under a probability measure $\widetilde{\mathbb{P}}$, one has $\mathbb{E}[1_{[Y>y]}] = \widetilde{\mathbb{P}}(Y > y)$.

Problem 6. Consider a bond market with zero coupon bonds of all possible maturities with their prices denotes by $P(t,T)$. Let $t \leq S < T$ be three fixed time instances.

Provide answers to the following three questions.

(i) Give the definition of the simple forward rate for the period $[S,T]$ contracted at t (the LIBOR forward rate $L(t;S;T)$).

(ii) Give the definition of the continuously compounded forward rate $R(t;S;T)$ for the period $[S,T]$ contracted at t.

(iii) Give the definition of the continuously compounded spot rate $R(S;T)$ for the period $[S,T]$.

Problem 7. A forward rate agreement (FRA) is a contract entered into at time $t = 0$ where two parties (a lender and a borrower) agree to let a certain interest rate, R^*, act on a prespecified principal amount, K, over some future period $[S,T]$. Assuming that the interest rate R^* is compounded continuously, the cash flow to the lender is by definition given as follows:

- At time S: $-K$.
- At time T: $Ke^{R^*(T-S)}$.

Provide answers to the following two questions.

(i) The above cash flow can be replicated in terms of zero coupon bonds bought at time $t = 0$. Show how this can be done.

(ii) By a no-arbitrage argument the value of the replicating portfolio consisting of zero coupon bonds that you have constructed in step (i) gives the value of the FRA at time $t = 0$. Show that in order for this value to be equal to zero, the rate R^* has to be equal to the forward rate $R(0;S;T)$.