Afdeling Wiskunde	Exam Stochastic Processes for Finance
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Give clear, but brief motivations for all your answers.
Calculators are not allowed.
Grade is total credits divided by 10.

1. (20=5+10+5 points.) Let $X_1, X_2, \dots$ be a sequence of independent random variables with $\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = 0) = 1/2$, for every i. For given $\mu > 0$, set

$$M_n = \prod_{i=1}^{n} (X_i + \mu)^2.$$

   a. Show that the filtrations generated by $X_1, X_2, \dots$ and $M_1, M_2, \dots$ are the same.
   b. For what value(s) of μ is the sequence $M_1, M_2, \dots$ a martingale (relative to its own filtration)?
   c. For what value(s) of μ is the sequence $M_1, M_2, \dots$ a Markov process (relative to its own filtration)?

2. (20=8+5+7 points.) Consider a (discrete time) market that consists of a bank account with fixed interest rate r and a stock with price S_n at time n, for S_n following the binomial tree model:

$$S_0 = 1,$$
$$\mathrm{P}(S_{n+1} = uS_n \mid S_0, \dots, S_n) = p, \qquad n \geq 0,$$
$$\mathrm{P}(S_{n+1} = dS_n \mid S_0, \dots, S_n) = 1 - p,$$

   where d and u are known numbers satisfying $0 < d < e^r < u$. Consider the option that pays S_N money units at the fixed time N if the stockprice S_N is strictly below a specified level K, and pays 0 otherwise.
   a. Characterize the stock price process in a "risk-neutral" market.
   b. Write S_N in terms of X_N, defined as the number of times $n \in \{0, 1, \dots, N-1\}$ that $S_{n+1} = uS_n$. What is the distribution of X_N in the "risk-neutral" market?
   c. Give an explicit formula (may be complicated) for the arbitrage-free price of the option at time 0.

3. (20=4+6+5+5 points.)
   a. Give the definition of a Brownian motion W.
   b. Derive a stochastic differential equation for the process $X_t = W_t^3 + \alpha W_t^2 + \beta W_t + \gamma t^2 + \delta \int_0^t W_s \, ds$.
   c. For which values $\alpha, \beta, \gamma, \delta$ is the process X a martingale?
   d. Show that for fixed $a, b > 0$ the process Y defined by $Y_t = (W_{(at+b)} - W_b)/\sqrt{a}$ is also a Brownian motion.

4. (20=5+5+5+5 points.) Consider a Black-Scholes type market consisting of a riskless bank account with value $R_t = e^{rt}$ and a stock with price $S_t = e^{\mu t + \sigma W_t}$, for W a Brownian motion.
   a. Derive an SDE for the discounted stock price process.
   b. Which process must be a martingale under the risk-neutral measure?
   c. Characterize the distribution of the stock price S_t at time t under the risk-neutral (i.e. martingale) measure for the market.
   d. Give an explicit formula for the price at time 0 of the option that pays the amount S_T at a prespecified time T if the stock price S_T is below a prespecified level K and pays 0 otherwise. [The formula may contain an integral, but no unknowns other than T, K, μ, σ and r.]

5. (20=5+5+5+5 points.)

a. Give the definition of a self-financing portfolio in a Black-Scholes market with a bank account with value $R_t = e^{rt}$ and a stock with price S_t.

b. Give the definition of the short rate.

c. Give the Hull-White model for the short rate.

d. Give the definition of VaR.