Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	December 15, 2009

Give your answers in English.
It is not allowed to use calculators, books or notes.
Good luck!

1. (Arbitrage arguments)

   (a) (*3 points*) Consider a forward contract corresponding to an agreement to buy an asset on a specified future date T for a specified price K. Assume that the value process of the asset is given by $(S_t, t \geq 0)$, and let r be the (constant) risk-free interest rate. Derive the fair value of the strike price K in terms of S_0 and r. (Hint: You can use the "usual" pricing formula.)

   (b) (*2 points*) Suppose that in the same market there are two different interest rates, r_t and q_t. Show that, if $r_t > q_t$ for all $0 \leq t \leq T$, there are arbitrage opportunities.

2. (Binomial tree model)

   (*4 points*) Consider an N-period model with $S_0 = s_0$, parameters u, d, interest rate r, and "real-world" probability measure defined by $\mathbb{P}(S_{n+1} = uS_n) = p = 1-\mathbb{P}(S_{n+1} = dS_n)$ for $n = 0, \ldots, N-1$. Derive the martingale measure $\mathbb{Q}$.

3. (Ito's formula)

   Let W be a Brownian motion and $a \neq 0$ a real number.

   (a) (*3 points*) Let $X_t = \exp\left(aW_t - \frac{bt}{2}\right)$. Determine b (as a function of a) so that the process $(X_t, t \geq 0)$ is a martingale.

   (b) (*3 points*) Determine for which functions f the process Y defined by $Y_t = W_t^3 + f(t)W_t$ is a martingale.

4. (Stochastic integration)

   Let W be a Brownian motion relative to the filtration $(\mathcal{F}_t, t \geq 0)$.

   (a) (*3 points*) Compute the integral $\int_0^t (W_s^2 - s)dW_s$.

   (b) (*3 points*) Let X be the process defined by $X_t = \int_0^t s dW_s$ for $t \geq 0$. Let $T > 0$; what are the mean and variance of X_T? Express $\mathbb{E}(X_t|\mathcal{F}_s)$ in terms of X for all $0 \leq s < t$ and motivate your answer.

5. (Black-Scholes model)

   (*5 points*) Let S and B denote the stock and bond price processes in a Black-Scholes market. Assume that the risk-free interest rate is a constant r. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Let a, b, c be three positive real numbers, and suppose that $\varphi_t = aS_t + b$ and $\psi_0 = c$. Determine ψ_t in such a way that the portfolio becomes self-financing.

6. (Silly stock market model)

   Consider a market in which a stock is traded with price process $S_t = W_t$, where W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$. Assume that the risk-free interest rate is zero. Let $T > 0$ and consider a derivative whose payoff at time T is $C = S_T^2 - T$.

   (a) (*2 points*) Show that the process X defined by $X_t = W_t^2 - t$ is a $\mathbb{P}$-martingale.

   (b) (*2 points*) Give an explicit expression for the price V_t of the derivative at time $t \leq T$.

7. (Short rate model)

   Consider a short rate model that gives the prices of T-bonds as functions of the short rate r, $P(t,T) = F^T(t, r_t)$, with the dynamics of the short rate modelled by the SDE

   $$dr_t = \mu dt + \sigma dW_t,$$

   where μ and σ are constants and W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$.

   (a) (*3 points*) Use Ito's formula to obtain an expression for $dP(t,T)$ in terms of F^T's derivatives and the parameters μ and σ.

   (b) (*3 points*) Assume that the function $F^T(t,x)$ satisfies the PDE

   $$F_t^T + (\mu - \lambda\sigma)F_x^T + \frac{1}{2}\sigma^2 F_{xx}^T - xF^T = 0$$

   for some λ, with $F^T(T,x) = 1$ for all x. Let $\tilde{P}(t,T) = B_t^{-1}P(t,T)$ be the discounted T-bond price at time $t \leq T$. Show that

   $$d\tilde{P}(t,T) = \frac{\sigma F_r^T}{B_t} d\tilde{W}_t,$$

   where

   $$\tilde{W}_t = W_t + \int_0^t \lambda ds.$$