Department of Mathematics	Exam "Stochastic Processes for Finance"
Vrije Universiteit	December 22, 2006

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a floating rate bond with face (principal) value 1, maturity T, and intermediate payments C_i at times $T_1 < T_2 < \ldots < T_n = T$ with $C_i = 1/P(T_{i-1}, T_i) - 1$. Use an arbitrage argument to price the floating rate bond by constructing a replicating portfolio.

2. (Discrete-time martingales)

   In this exercise time is discrete, $(\mathcal{F}_n)$ is a given filtration. Consider a discrete-time process X such that $X_{n+1} = f(X_n)$ for some deterministic function f. Show that X is a martingale if and only if f is the identity.

3. (Binomial model)

   Consider a process $S = (S_0, \ldots, S_n)$ which follows an n-period binomial model. The price S_0 at time 0 is a given number and at each time j the next value S_{j+1} is uS_j or dS_j with probabilities p and $1-p$, respectively, where $p \in (0,1)$ and $d < 1 < u$ are given constants. Let $(\mathcal{F}_j)$ be the natural filtration of S.

   (a) Explain why the binomial model is Markovian.

   (b) Give an expression for the conditional expectation $\mathbb{E}_p(S_{j+1} \,|\, S_j)$.

   (c) For which value(s) of p is S a martingale with respect to the filtration $(\mathcal{F}_j)$, and why?

4. (Brownian motion)

   Let W be a Brownian motion and denote by $(\mathcal{F}_t)$ its natural filtration.

   (a) Using Itô's formula, show that the process X defined by $X_t = W_t^3 - ctW_t$ is a martingale with respect to $(\mathcal{F}_t)$ if and only if $c = 1$.

   (b) For which values of parameters a and b is the process Y defined by $Y_t = W_{at}^2 - bt$ a martingale?

5. (Stochastic calculus)

   Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of the interval $(0, T]$ and let f be a simple (i.e., piecewise constant) function: $f(t) = c_i$, $t \in (t_i, t_{i+1}]$, for $i = 0, \ldots, n-1$. Show that

(a) $\mathbb{E}\int_0^T f(t)\,dW_t = 0$

(b) $\mathbb{E}(\int_0^T f(t)\,dW_t)^2 = \mathbb{E}\int_0^T f^2(t)\,dt$

(c) $\mathbb{E}(\int_0^T f(t)\,dW_t\,|\,\mathcal{F}_S) = \int_0^S f(t)\,dW_t,\ \forall S \le T$

6. (Black-Scholes)

Let $D_t = e^{qt}$ and $B_t = e^{rt}$ be the prices of US dollar and euro bonds respectively, with q the US interest rate and r the European interest rate. Let the exchange rate E_t, i.e. the euro value of one dollar, be modelled by a geometric Brownian motion,

$$E_t = E_0 e^{\nu t + \sigma W_t}.$$

A European investor can trade in two assets: the risk-less euro bond B and the "risky" US bond $S = ED$. Consider a contract giving a European investor the right to buy one US dollar for K euros at some specified future time $T > 0$. Express the fair price of this contract in terms of the price of a European call option in the standard Black-Scholes model. Specify the strike price and maturity of the call option.

7. (Short rate model)

Consider a term structure model that gives the prices of T-bonds as functions of the short rate r, $P(t,T) = F^T(t, r_t)$, with the dynamics of the short rate modelled by the SDE

$$dr_t = \mu dt + \sigma dW_t,$$

where W is a Brownian motion under the "real-world" probability measure $\mathbb{P}$.

(a) Use Ito's formula to obtain an expression for $dP(t,T)$ in terms of F^T, its derivatives, and the parameters μ and σ.

(b) Let $\mathbb{Q}$ be the probability measure under which $\tilde{P}(t,T) = B_t^{-1}P(t,T)$ is a martingale. Under $\mathbb{Q}$, the short rate satisfies the following SDE

$$dr_t = (\mu - \sigma\lambda)dt + \sigma d\tilde{W}_t, \tag{1}$$

where λ is the market price of risk and $\tilde{W}$ is a $\mathbb{Q}$-Brownian motion. Use (a) and equation (1) to show that

$$\tilde{W}_t = W_t + \int_0^t \lambda\,ds.$$

Norming:

1:	3	2:	3	3(a):	1	4(a):	4	5(a):	2	6:	5	7(a):	4
				3(b):	1	4(b):	4	5(b):	2			7(b):	4
				3(c):	1			5(c):	2				

Grade = (total+4)/4