Department of Mathematics	Exam “Stochastic Processes for Finance”
Vrije Universiteit	February 2, 2006

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a world in which a stock is traded which has price S_t at time t and in which there is a bank with fixed interest rate r, i.e. 1 euro at time t grows to $\exp(rt)$ euros at time t. Money can be borrowed at the same rate.

   In this world, consider a derivative which at a predetermined time T in the future, pays to the owner the amount S_T if $S_T \leq K$ and $2K - S_T$ if $S_T > K$, where K is a fixed number.

   (a) Construct a self-financing portfolio consisting of call options, put options and money in the bank, which replicates the pay-off of this derivative. (Specify the strike prices and expiry times of the options in the portfolio.)

   (b) Using the answer of part (a) and an arbitrage argument, express the value of the derivative at time t in terms of the prices of certain call and put options and the interest rate.

2. (Discrete-time martingales)

   In this exercise time is discrete, $(\mathcal{F}_n)$ is a given filtration. Show that if X is a predictable process and M is a martingale, then the process Y defined by

$$Y_n = \sum_{k=1}^{n} X_k(M_k - M_{k-1})$$

   is a martingale as well.

3. (Random walk)

   Let S be the simple random walk. This means that $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, where the X_i's are independent and $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$.

   (a) Show that the process S is a martingale with respect to its natural filtration.

   (b) Show that the process $M_n = S_n^2 - n$ is a martingale with respect to the natural filtration of S.

4. (Brownian motion)

   Let W be a Brownian motion and a a real number. Denote the natural filtration of W by $(\mathcal{F}_t)$.

(a) Show that $(W_{t+1} - W_1)_{t \geq 0}$ is a Brownian motion.

(b) Using Itô's formula, show that the process X defined by $X_t = W_t^2 - at$ is martingale with respect to $(\mathcal{F}_t)$ if and only if $a = 1$.

(c) Consider $Y_t = f(t, W_t)$ for a smooth function f. What partial differential equation must f satisfy for Y to be a martingale?

5. (Black-Scholes)

Let B and S be the bond and stock price processes in a Black-Scholes market. These are assumed to satisfy $B_t = \exp(rt)$ and $S_t = \exp(\mu t + \sigma W_t)$, where $r > 0$, μ and σ are given numbers and W is a Brownian motion. Consider a portfolio (φ, ψ), where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Suppose that $\varphi_t = \int_0^t S_u \, du$ and $\psi_0 = 0$. Determine ψ_t in such a way that the portfolio becomes self-financing.

6. (Silly stock market model)

Consider a world in which a stock is traded with price process $S_t = W_t + t$, where W is a Brownian motion under the real-world probability measure $\mathbb{P}$, and with a bank with zero interest. Let $T > 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t.

(a) Explain why the the process S_t is a Brownian motion under the martingale measure $\mathbb{Q}$.

(b) Give an integral expression for the price V_0 of the derivative at time 0.

7. (Hull-White)

In the Hull-White model the short rate is assumed to satisfy, under the martingale measure $\mathbb{Q}$, the SDE

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma \, dW_t,$$

where W is a Brownian motion, a, σ are constants and θ is a deterministic function.

(a) Apply Itô's formula to $\exp(at)r_t$ to express $\exp(at)r_t - r_0$ as the sum of a stochastic integral and an ordinary integral.

(b) Using the answer of part (a), determine, for a fixed $t \geq 0$, the distribution of r_t under $\mathbb{Q}$.

Norming:

1(a):	2	2:	3	3(a):	2	4(a):	2	5 :	5	6(a):	3	7(a):	3
1(b):	1			3(b):	2	4(b):	3			6(b):	4	7(b):	2
						4(c):	4						

Grade = (total+4)/4