Department of Mathematics	Practice exercises “Stochastic Processes for Finance”
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Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

   Consider a standard Black-Scholes market with fixed interest rate r, let S_t denote the stockprice at time $t \geq 0$. Fix a maturity $T > 0$, strike price $K > 0$ and denote by C_t, resp. P_t, the value at time $t \leq T$ of a European call option, resp. put option, with maturity T and strike price K. Using an arbitrage argument, show that

   $$C_t - P_t = S_t - Ke^{-r(T-t)}$$

   for all $t \in [0, T]$ (this is the so-called *call-put parity*).

2. (Discrete-time martingales)

   In this exercise time is discrete, $(\mathcal{F}_n)$ is a given filtration. Show that if a (discrete-time) process X is predictable and also a martingale, it is constant, i.e. $X_n = X_0$ for all n.

3. (Random walk)

   Let $p \in (0, 1)$ be given. Define the discrete-time process S by putting $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, where the X_i's are independent and $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = p$. Show that S is a martingale with respect to its natural filtration if and only $p = 1/2$.

4. (Brownian motion)

   (a) Let W be a Brownian motion. Show that the processes $-W$ and $(W_{t+1} - W_1)_{t\geq 0}$ are Brownian motions as well.

   (b) Let Z be standard normally distributed and define $X_t = \sqrt{t}Z$. Is the process X a Brownian motion? (Explain why/why not.)

5. (Stochastic calculus)

   Let W be a Brownian motion and $(\mathcal{F}_t)$ its natural filtration.

   (a) Using Itô's formula, show that the process

   $$\Big(\int_0^t W_s \, ds - tW_t \Big)_{t\geq 0}$$

   is a martingale.

(b) Using part (a) and Itô's formula, show that the process

$$(W_t^3 - 3tW_t)_{t\geq 0}$$

is a martingale.

6. (Black-Scholes)

   Let B and S be the bond and stock price processes in a Black-Scholes market. These are assumed to satisfy $B_t = \exp(rt)$ and $S_t = \exp(\mu t + \sigma W_t)$, where r, μ and σ are given numbers and W is a Brownian motion.

   (a) Use Itô's formula to obtain a stochastic differential equation for the process S.

   (b) Determine the quadratic variation process $[S]$.

7. (Hull-White)

   In the Hull-White model the short rate is assumed to satisfy, under the martingale measure $\mathbb{Q}$, the SDE

   $$dr_t = (\theta(t) - ar_t)\,dt + \sigma\,dW_t,$$

   where W is a Brownian motion, a, σ are constants and θ is a deterministic function. Apply Itô's formula to $\exp(at)r_t$ to express $\exp(at)r_t - r_0$ as the sum of a stochastic integral and an ordinary integral.