Short answers to the "Stochastic Processes for Finance" exam December 23, 2005

1. (a) Consider a portfolio consisting of one call and one put option with strike $K$ and maturity $T$. This portfolio is clearly self-financing and its pay-off equals $(S_T - K)_+ + (K - S_T)_+ = |S_T - K|$, hence it replicates the derivative.

   (b) Let $C_t$ (resp. $P_t$) be the value at time $t$ of the call (resp. put) option with strike $K$ and maturity $T$. Let $V_t$ be the value of the derivative at time $t$. The portfolio of part (a) is self-financing and its value at time $T$ equals the value of the derivative. Hence, by absence of arbitrage, the value of the portfolio and the derivative should be equal at all times. We conclude that $V_t = C_t + P_t$ for $t \in [0, T]$.

2. Fix $n \in \mathbb{N}$. The fact that $X$ is a martingale implies that $\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = X_n$. On the other hand, predictability means that given the information in $\mathcal{F}_n$, $X_{n+1}$ is known, so that $\mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) = X_{n+1}$. Combining the two identities we find that $X_{n+1} = X_n$. By induction, it follows that $X_n = X_0$ for all $n$.

3. Consider a process $S = (S_0, \ldots, S_n)$ which follows an $n$-period binomial model. The price $S_0$ at time $t = 0$ is a given number and at each time $t$ the next value $S_{t+1}$ is $uS_t$ or $dS_t$ with probabilities $p$ and $1 - p$, respectively, where $p \in (0, 1)$ and $d < 1 < u$ are given constants. Let $(\mathcal{F}_t)$ be the natural filtration of $S$.

   (a) Given $\mathcal{F}_t$, $S_{t+1}$ is $uS_t$ or $dS_t$ with probabilities $p$ and $1-p$, respectively. Hence

   $$\mathbb{E}_p(S_{t+1} \,|\, \mathcal{F}_t) = puS_t + (1 - p)dS_t = (p(u - d) + d)S_t.$$

   (b) The process is a martingale if and only if $\mathbb{E}_p(S_{t+1} \,|\, \mathcal{F}_t) = S_t$ for all $t$. In view of (a) this holds if and only if $p = (1 - d)/(u - d)$.

4. Let $W$ be a Brownian motion and $a$ a positive number. Denote the natural filtration of $W$ by $(\mathcal{F}_t)$.

   (a) The can be shown using the definition of the BM, using the observation that the natural filtration of $X$ is $(\mathcal{F}_{at})_t$.

   (b) By Itô's formula, $dW_t^3 = 3W_t^2\, dW_t + 3W_t\, dt$ and $d(tW_t) = t\, dW_t + W_t\, dt$. Combining this gives

   $$dY_t = (3W_t^2 - 3t)\, dW_t.$$

   So $Y$ is an integral with respect to BM, hence a martingale.

   (c) We have $Z_t = f(X_t)$, with $f(x) = \exp(x)$ and $X_t = aW_t - a^2t/2$. An application of Itô's formula gives (check!) the SDE

   $$dZ_t = aZ_t\, dW_t.$$

   In particular $Z$ is an integral wrt BM and hence a martingale.

5. Let $V_t = \varphi_t S_t + \psi_t B_t$ be the value process of the portfolio. Then, since $\varphi_t = S_t$, we have $V_t = S_t^2 + \psi_t B_t$ and hence

$$dV_t = 2S_t\, dS_t + d[S]_t + \psi_t\, dB_t + B_t\, d\psi_t.$$

On the other hand we want the portfolio to be self-financing, which means, by definition, that

$$dV_t = \varphi_t\, dS_t + \psi_t\, dB_t = S_t\, dS_t + \psi_t\, dB_t.$$

Comparing the two SDE's yields

$$B_t\, d\psi_t = -S_t\, dS_t - d[S]_t.$$

Dividing by $B_t$ and using that $\psi_0 = 0$ we find

$$\psi_t = -\int_0^t B_u^{-1} S_u\, dS_u - \int_0^t B_u^{-1}\, d[S]_u.$$

6. (a) In general models we have that $V_0$ is the expectation of the discounted pay-off under the martingale measure. In this case the martingale measure $\mathbb{Q}$ equals the ordinary measure $\mathbb{P}$ and there is no interest, hence no discounting. So, since $W_T \sim N(0,T)$ under $\mathbb{P}$,

$$V_0 = \mathbb{E}_{\mathbb{P}} C = \mathbb{E}_{\mathbb{P}} f(W_T) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi T}} f(x) e^{-\frac{x^2}{2T}}\, dx.$$

   (b) We have that $V_t = \mathbb{E}_{\mathbb{Q}}(e^{-rt}C \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(C \,|\, \mathcal{F}_t)$, hence $V$ is a $\mathbb{P}$-martingale. On the other hand, Itô's formula gives

$$dV_t = F_t(t, S_t)\, dt + F_x(t, S_t)\, dW_t + \tfrac{1}{2} F_{xx}(t, S_t)\, dt.$$

   Then $V$ can only be a martingale if the $dt$-terms vanish. This yields the PDE $F_t + (1/2)F_{xx} = 0$.

7. (a) See the proof of Lemma 8.12 in the lecture notes:

$$e^{at} r_t - r_0 = \int_0^t \theta(u) e^{au}\, du + \sigma \int_0^t e^{au}\, dW_u.$$

   (b) Use (a), the fact that the integral of a deterministic integrand wrt BM is Gaussian and the formula for the second moment of a stochastic integral to see that

$$e^{at} r_t \sim N(m, s^2),$$

   with

$$m = r_0 + \int_0^t \theta(u) e^{au}\, du, \qquad s^2 = \sigma^2 \int_0^t e^{2au}\, du.$$

   It follows that

$$r_t \sim N(e^{-at} m, e^{-2at} s^2).$$