Resit Stochastic Modelling (X_400646)

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Faculty of Science

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This exam consists of five exercises, for which you can obtain 54 points in total. Your grade will be calculated as (number of points + 6)/6. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5\}$.

(a) [5pt] Assume the transition matrix is

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

For initial state (i) $X_0 = 1$, determine with which probability the Markov chain ends up in each of the absorbing classes. For initial states (ii) $X_0 = 2$, (iii) $X_0 = 5$, determine whether a limit distribution exists and find the limit distribution in case it exists.

(b) [4pt] Assume the transition matrix is

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(question continued on next page)

Note that the only difference with part (a) is in the transitions out of state 3. What is the expected number of steps it takes to reach state 4 from state 1?

Question 2. Every day Bob commutes to work in the morning and then commutes back home in the evening. From time to time he likes to buy a coffee to-go for his commute. As an environmentally conscious person, Bob owns three travel coffee cups and uses them for his to-go coffees when he can.

To be more specific, each commute Bob feels like having a coffee with probability 2/3, independently of his other commutes. If Bob does feel like having a coffee and finds one of the travel cups at the starting location of the commute, he grabs that cup along (so the cup transfers to the end location of the commute). If Bob does feel like having a coffee but finds no travel cup at the starting location, he will buy a coffee in a disposable cup. If Bob does not feel like having a coffee, he does not carry any travel cups along.

(a) [**5pt**] Argue that the following sequence is a discrete-time Markov chain:

$$X_n = \text{number of travel cups at the } \textit{starting location of commute } n.$$

Hint: note that the starting location of the next commute $n+1$ is the end location of commute n. Hence, you need to consider transitions between the starting and end locations of a commute.

(b) [**4pt**] What is the long-run fraction of commutes for which Bob finds no travel cup at the starting location?

(c) [**3pt**] Each time Bob uses a travel coffee cup, he gets a discount of €0.25 on his coffee. How much does Bob save on average per commute over the long time-run?

Question 3. An emergency desk receives true alarms according to a Poisson process with rate 8 per day. It also receives false alarms according to a Poisson process with rate 1 per day. The two Poisson processes are independent.

(a) [**4pt**] What is the probability that the next two alarms are both true alarms?
Hint: what is the distribution of the times between successive true alarms? and between successive false alarms?

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(b) [6pt] A 24-hour day consists of three 8-hour shifts. What is the probability that exactly 3 alarms (in total, true and false) are received on a given day but none of them is received in the middle shift of the day?

Question 4. Consider a single-server system where *potential* customers arrive according to a Poisson process with rate λ. If, upon arrival, a customer finds $i = 1, 2, \ldots$ other customers in the system, then this customer joins the queue with probability $1/(i+1)$ or leaves immediately with probability $i/(i+1)$. A customer that arrives into an empty system immediately proceeds to the server. The service times are distributed exponentially with rate μ.

(a) [5pt] Argue that the number of customers in the system is a continuous-time Markov chain. Intuitively, for which λ and μ is this system stable?

(b) [4pt] Find the occupancy distribution p^{occ} in the stable scenario.
Hint: the Taylor expansion for the exponential function is $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

(c) [3pt] Express the fraction of lost customers in terms of the probabilities p_i^{occ}. You do not have to further plug in the formulas for p^{occ} that you found in (b).

Question 5. Jobs arrive at a server according to a Poisson process of rate $\lambda = 1/3$ and are served in the order of arrival. For 1/4 of the jobs, their service times have a normal $N(4, 1^2)$ distribution. The remaining 3/4 of the jobs require a fixed service time 2.

(a) [5pt] Find the average waiting time EW.

For part (b), consider a new situation where the server requires *start-up times* of fixed duration 3. That is, when, after an idling period with no jobs to do, the server receives a job, it will only start serving that job 3 time units later.
You can use without proof the fact that the system with such start-up times is still stable, and hence the long-run fraction of time that the server is busy serving jobs is given by ρ. Moreover, denote by Π_{idle} the long-run fraction of time that the server is idling with no jobs to do, and denote by $\Pi_{\text{start-up}}$ the long-run fraction of time that the server is starting-up.

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(b) [6pt] Find the average waiting time EW in this new situation by doing Mean Value Analysis.
Hint: in the arrival relation, consider what an arriving job has to wait for if it arrives when the server is idling with no jobs to do, when the server is starting-up, and when the server is busy serving another job. You can leave Π_{idle} and $\Pi_{\text{start-up}}$ as they are in your solution, you do not have to calculate them.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2) E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$