Final exam Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

December 20, 2023, 12:15–14:30

This exam consists of three exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points $+ 5)/5$. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Two auto mechanics, a senior one and a junior one, do car repairs in a three-car garage. Cars drop by this garage according to a Poisson process at a rate 3 cars per day. If a car drops by a full garage, it leaves immediately to seek service elsewhere. If a car drops by an empty garage, it is handled by the junior mechanic. If a car drops by while one of the mechanics is vacant, it is handled by that mechanic. If a car drops by while both mechanics are busy but the third spot in the garage is available, that car is taken into the garage and, as soon as one of the mechanics becomes vacant, it will be handled by that mechanic. The repair times are distributed exponentially, with rate 2 cars per day for the senior mechanic and 1 car per day for the junior mechanic.

(a) [4pt] Formulate a CTMC based on which you can answer the subsequent parts of the question.
Hint: When there is a single car in the garage, is additional information required to determine the next state of the CTMC?

(b) [5pt] What fraction of time is (i) the senior mechanic vacant, (ii) the junior mechanic vacant? You can use without derivation the fact that the garage is full for $\frac{5}{17}$ of the time.

(c) [2pt] What fraction of cars dropping by this garage (i) leave without repair, (ii) experience waiting before repair?

(d) [2pt] What is the time-average number of cars in this garage?

Question 2. Consider an $M/M/3$ system with *impatient* customers. The rate of *arrival attempts* is λ. An arriving customer that would have to wait to get service joins the queue with probability α or immediately leaves the system with probability $1-\alpha$. The service rate is μ at each of the servers.

(a) [**4pt**] Argue that the number of customers in this system is a CTMC. Argue intuitively for which λ, α and μ this CTMC is stable.

(b) [**4pt**] Find the occupancy distribution p^{occ}. In particular derive that

$$p_0^{occ} = \left[1 + \lambda/\mu + \frac{(\lambda/\mu)^2}{2} + \frac{(\lambda/\mu)^3}{6}\frac{1}{1-\dfrac{\alpha\lambda}{3\mu}}\right]^{-1}.$$

Hint: For normalisation, group states $i \geq 3$.

(c) [**4pt**] Denote by Π_{loss} the fraction of customers lost due to impatience. Denote by Π_W the fraction of customers that experience waiting *out of those customers that join the system.* Knowing the occupancy distribution, how can you find Π_{loss} and Π_W?
Remark: Expressions in terms of p^{occ} are sufficient; you do not have to further plug in the formulas for p^{occ} that you found in (b).

(d) [**4pt**] Find the time-average number of customers waiting in the queue by doing Mean Value Analysis.
Remark: If you need to make use of Π_{loss} and Π_W as defined in (c), you can use them as they are, without plugging in the formulas you found for them in (c).

(e) [**3pt**] Assume the arrival process is still Poisson with rate λ but the service time distribution has changed to non-exponential while the mean service time remains the same, $1/\mu$. Does the system still have the occupancy distribution that you found in (b) in case (i) $\alpha = 0$ (no customers are willing to wait), (ii) $\alpha = 1$ (all customers are willing to wait)?

Question 3. Consider two data transmission channels between the same source and destination. At present, the two channels do not interact but each channel handles a specific type of packages on its own. Channel 1 receives type-1 packages according to a Poisson process of rate λ_1 and transmits them in the FIFO order; their transmission times are distributed as B_1.

Channel 2 receives type-2 packages according to a Poisson process of rate λ_2 and transmits them in the FIFO order as well; their transmission times are distributed as B_2. It needs investigation whether it would be beneficial to merge the two channels. The two channels are not of equal capacities and, under merging, the transmission times would become $\frac{1}{4}B_1$ for type-1 packages and $\frac{3}{4}B_2$ for type-2 packages.

It is known that $\lambda_1 = 0.2$, $\mathbb{E}B_1 = 4$, $\mathbb{V}B_1 = 1$, $\lambda_2 = 0.1$, $\mathbb{E}B_2 = 8$, $\mathbb{V}B_2 = 4$ (where $\mathbb{V}$ stands for the variance). The average *transmission delay* (i.e., the time between the moment the package is received at the channel and the moment its transmission starts) is presently 8.5 at channel 1 and 17 at channel 2.

(a) [**4pt**] What is going to be the average transmission delay at the merger channel if it transmits in the FIFO order?
Hint: Be reminded that, for a random variable X and constant a, $\mathbb{E}((aX)^2) = a^2\mathbb{E}(X^2)$.

(b) [**4pt**] Would the merging be beneficial for the transmission times, for the transmission delays, and for the total time a package spends at the transmission channel?

(c) [**2pt**] Would your conclusions from (b) still hold if, instead of transmission in the FIFO order, the next message to transmit is selected at random from the queue?

(d) [**3pt**] For channel 1 before merging, find the expected stretch of time during which the channel is continuously transmitting packages (between two periods of having no packages to transmit).

To scan your exam,

- *if you are a regular-time student*
    - *and done before 14:25, walk to the front row for scanning,*
    - *and done after 14:25, remain at your seat and wait for an announcement to start scanning,*
- *if you are an extra-time student, walk to the front row for scanning.*

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho} (1 + c_B^2) E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1} (c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$