SOLUTIONS

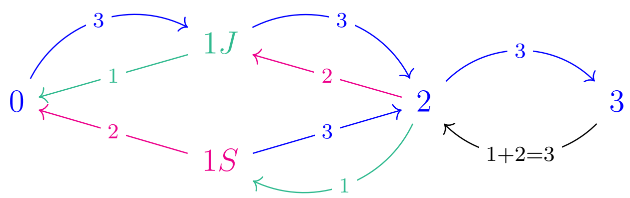
Final exam Stochastic Modelling

December 20, 2023

Question 1. Two auto mechanics, a senior one and a junior one, do car repairs in a three-car garage. Cars drop by this garage according to a Poisson process at a rate 3 cars per day. If a car drops by a full garage, it leaves immediately to seek service elsewhere. If a car drops by an empty garage, it is handled by the junior mechanic. If a car drops by while one of the mechanics is vacant, it is handled by that mechanic. If a car drops by while both mechanics are busy but the third spot in the garage is available, that car is taken into the garage and, as soon as one of the mechanics becomes vacant, it will be handled by that mechanic. The repair times are distributed exponentially, with rate 2 cars per day for the senior mechanic and 1 car per day for the junior mechanic.

(a) [4pt] Formulate a CTMC based on which you can answer the subsequent parts of the question.
Hint: When there is a single car in the garage, is additional information required to determine the next state of the CTMC?

Solution Let $X(t)$ be the number of cars in the garage at time t if that number is different than 1. In case of exactly 1 car in the garage at time t, let $X(t)$ also specify which mechanic is handling that car (J standing for junior and S for senior). Then $X(t)$ is a CTMC with the transition diagram



(b) [5pt] What fraction of time is (i) the senior mechanic vacant, (ii) the junior mechanic vacant? You can use without derivation the fact that the garage is full for $\frac{5}{17}$ of the time.

Solution Since $X(\cdot)$ is an irreducible CTMC on a finite state space, the occupancy distribution exists and solves the balance and normalization equations (we use balance per state):

$$\begin{cases} p_0 * 3 = p_{1J} * 1 + p_{1S} * 2, & 4)\ 3p_0 = \frac{4}{5}p_3 + \frac{2}{5}p_3 = \frac{6}{5}p_3 \Rightarrow p_0 = \frac{2}{5}p_3 \\ p_{1J} * 4 = p_0 * 3 + p_2 * 2, & 3) \text{ plug in the above equation} \Rightarrow 4p_{1J} = p_{1J} + 2p_{1S} + 2p_2 \\ & \Rightarrow 3p_{1J} = \frac{2}{5}p_3 + 2p_3 = \frac{12}{5}p_3 \Rightarrow p_{1J} = \frac{4}{5}p_3 \\ p_{1S} * 5 = p_2 * 1, & 2)\ p_{1S} = \frac{1}{5}p_2 = \frac{1}{5}p_3 \\ p_2 * 4 = p_{1J} * 3 + p_{1S} * 3 + p_3 * 3, & \\ p_3 * 3 = p_2 * 3 & 1)\ p_2 = p_3 \\ p_0 + p_{1J} + p_{1S} + p_2 + p_3 = 1. & \end{cases}$$

It is given that $p_3^{occ} = \frac{5}{17}$, so we express all other occupancy probabilities via p_3^{occ} and get the full occupancy distribution

$$p_0^{occ} = \frac{2}{17}, \quad p_{1J}^{occ} = \frac{4}{17}, \quad p_{1S}^{occ} = \frac{1}{17}, \quad p_2^{occ} = \frac{5}{17}, \quad p_3^{occ} = \frac{5}{17}$$

To answer the questions,
(i) the fraction of time the senior server is vacant is $p_0^{occ} + p_{1J}^{occ} = \frac{6}{17}$,
(ii) the fraction of time the junior server is vacant is $p_0^{occ} + p_{1S}^{occ} = \frac{3}{17}$.

(c) [2pt] What fraction of cars dropping by this garage (i) leave without repair, (ii) experience waiting before repair?

Solution In (i), we want the fraction of cars that pass by and see a full garage. By PASTA, it is $p_3^{occ} = \frac{5}{17}$.
In (ii), we want the fraction of cars that pass by and see 2 cars in the garage. By PASTA, it is $p_2^{occ} = \frac{5}{17}$.

(d) [2pt] What is the time-average number of cars in this garage?

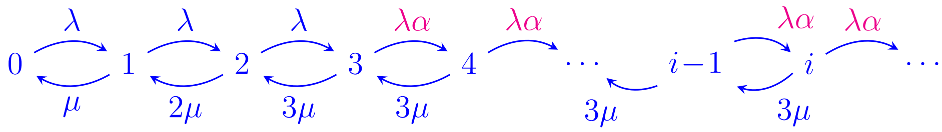
Solution It is

$$p_0^{occ} * 0 + (p_{1J}^{occ} + p_{1S}^{occ}) * 1 + p_2^{occ} * 2 + p_3^{occ} * 3 = \frac{4 + 1 + 5 * 2 + 5 * 3}{17} = \frac{30}{17}.$$

Question 2. Consider an $M/M/3$ system with *impatient* customers. The rate of *arrival attempts* is λ. An arriving customer that would have to wait to get service joins the queue with probability α or immediately leaves the system with probability $1 - \alpha$. The service rate is μ at each of the servers.

(a) **[4pt]** Argue that the number of customers in this system is a CTMC. Argue intuitively for which λ, α and μ this CTMC is stable.

Solution $L(t)$ = number of customers in the system at time t is a CTMC with the transition diagram



Intuitively, the system is stable if departures are faster than arrivals in large states, i.e. this system is stable if

$$\lambda\alpha < 3\mu.$$

(b) **[4pt]** Find the occupancy distribution p^{occ}. In particular derive that

$$p_0^{occ} = \left[1 + \lambda/\mu + \frac{(\lambda/\mu)^2}{2} + \frac{(\lambda/\mu)^3}{6}\frac{1}{1 - \dfrac{\alpha\lambda}{3\mu}}\right]^{-1}.$$

Hint: For normalisation, group states $i \geq 3$.

Solution Below we find a solution to balance and normalization equations. This solution is p^{occ} since $L(t)$ is an irreducible CTMC.

The system for p^{occ} is,

$$\begin{cases} \text{balance for set } \{0\} & p_0 * \lambda = p_1 * \mu, \\ \text{balance for set } \{0,1\} & p_1 * \lambda = p_2 * 2\mu, \\ \text{balance for set } \{0,1,2\} & p_2 * \lambda = p_3 * 3\mu, \\ \text{balance for set } \{0,\dots,i-1\} & p_{i-1} * \lambda\alpha = p_i * 3\mu, \quad i \geq 4, \\ \text{normalization} & \sum_{i=0}^{\infty} p_i = 1. \end{cases}$$

Hence,

$$p_1 = \frac{\lambda}{\mu}p_0, \quad p_2 = \frac{\lambda}{2\mu}p_1 = \frac{(\lambda/\mu)^2}{2}p_0, \quad p_3 = \frac{\lambda}{3\mu}p_2 = \frac{(\lambda/\mu)^3}{6}p_0,$$

and for $i \geq 4$,

$$p_i = \frac{\lambda\alpha}{3\mu}p_{i-1} = \left(\frac{\lambda\alpha}{3\mu}\right)^2 p_{i-2} = \ldots = \left(\frac{\lambda\alpha}{3\mu}\right)^{i-3} p_3 = \frac{(\lambda/\mu)^3}{6}\left(\frac{\lambda\alpha}{3\mu}\right)^{i-3} p_0 \quad \text{(also true for } i = 3).$$

Now we follow the hint for the normalization equation and get

$$1 = \sum_{i=0}^{\infty} p_i = \sum_{i=0}^{2} p_i + \sum_{i=3}^{\infty} p_i = p_0 \left(1 + \lambda/\mu + \frac{(\lambda/\mu)^2}{2} + \frac{(\lambda/\mu)^3}{6} \underbrace{\sum_{i=3}^{\infty} (\tfrac{\lambda\alpha}{3\mu})^{i-3}}_{=\sum_{j=0}^{\infty}(\frac{\lambda\alpha}{3\mu})^j = \frac{1}{1-\frac{\lambda\alpha}{3\mu}}} \right).$$

To summarize, the last derivation implies that p_0^{occ} is as given in the question, and

$$p_1^{occ} = \frac{\lambda}{\mu} p_0^{occ}, \quad p_2^{occ} = \frac{(\lambda/\mu)^2}{2} p_0^{occ},$$
$$\text{for } i \geq 3, \quad p_i^{occ} = \frac{(\lambda/\mu)^3}{6} \left(\frac{\lambda\alpha}{3\mu} \right)^{i-3} p_0.$$

(c) [**4pt**] Denote by Π_{loss} the fraction of customers lost due to impatience. Denote by Π_W the fraction of customers that experience waiting *out of those customers that join the system*. Knowing the occupancy distribution, how can you find Π_{loss} and Π_W?
Remark: Expressions in terms of p^{occ} are sufficient; you do not have to further plug in the formulas for p^{occ} that you found in (b).

Solution Fraction p_i^{occ} of arrival attempts see i customers in the system, by PASTA.

Lost due to impatience are those arrival attempts who would have to wait, because see $i \geq 3$ customers in the system, but do not wish to wait, with probability $1 - \alpha$. Hence,

$$\Pi_{\text{loss}} = \left(\sum_{i \geq 3} p_i^{occ} \right) (1 - \alpha) = (1 - \alpha)(1 - p_0^{occ} - p_1^{occ} - p_2^{occ}).$$

Customers that experience waiting are those who, upon arrival, see $i \geq 3$ customers in the system and are willing to wait, with probability α. *Customers that join the system are those who are not lost.* We should pick those who experience waiting *out of those who join the system*, i.e.

$$\Pi_W = \frac{\left(\sum_{i \geq 3} p_i^{occ}\right) \alpha}{1 - \Pi_{\text{loss}}} = \frac{\alpha(1 - p_0^{occ} - p_1^{occ} - p_2^{occ})}{1 - \Pi_{\text{loss}}}.$$

(d) [**4pt**] Find the time-average number of customers waiting in the queue by doing Mean Value Analysis.
Remark: If you need to make use of Π_{loss} and Π_W as defined in (c), you can use them as they are, without plugging in the formulas you found for them in (c).

Solution The MVA equations are:

$$\begin{cases} \text{Little's law} & \mathbb{E}L^q \quad = \lambda_{\text{eff}} * \mathbb{E}W = \lambda(1 - \Pi_{\text{loss}}) * \mathbb{E}W, \\ \text{arrival relation} & \mathbb{E}W \quad = \Pi_W * \dfrac{1}{3\mu} + \mathbb{E}L^q * \dfrac{1}{3\mu}. \end{cases}$$

In the Little's law, we need the effective arrival rate into the system, i.e., we need the arrival attempts that are not lost.

In the arrival relation, the logic is as follows:
- Proportion Π_W of customers that *join the system* will have to wait. First of all, an arrival that has to wait will wait for a service completion among the three residual service times at the servers. Due to the memorylessness, that will take an Exponential(3μ) amount of time, in expectation $\frac{1}{3\mu}$.
- A new arrival sees L^q customers in the system and, after the very first service completion discussed in the previous bullet, L^q more service completions are necessary for the new arrival to reach a free server. Each service completion, again, takes an Exponential(3μ) amount of time, in expectation $\frac{1}{3\mu}$.

We plug in the Little's law into the arrival relation and get

$$\mathbb{E}W = \Pi_W * \frac{1}{3\mu} + \lambda(1 - \Pi_{\text{loss}}) * \mathbb{E}W * \frac{1}{3\mu} \quad \Rightarrow \quad \mathbb{E}W = \frac{\frac{\Pi_W}{3\mu}}{1 - \frac{\lambda(1-\Pi_{\text{loss}})}{3\mu}}.$$

(e) [**3pt**] Assume the arrival process is still Poisson with rate λ but the service time distribution has changed to non-exponential while the mean service time remains the same, $1/\mu$. Does the system still have the occupancy distribution that you found in (b) in case (i) $\alpha = 0$ (no customers are willing to wait), (ii) $\alpha = 1$ (all customers are willing to wait)?

Solution In (i), we have an $M/G/c/c$ system, with $c = 3$. This system is insensitive to the service time distribution and has the same p^{occ} as found in (b).

In (ii), we have an $M/G/c$ system, with $c = 3$. This system is sensitive to the service time distribution and does not have the same p^{occ} as found in (b).

Question 3. Consider two data transmission channels between the same source and destination. At present, the two channels do not interact but each channel handles a specific type of packages on its own. Channel 1 receives type-1 packages according to a Poisson process of rate λ_1 and transmits them in the FIFO order; their transmission times are distributed as B_1. Channel 2 receives type-2 packages according to a Poisson process of rate λ_2 and transmits them in the FIFO order as well; their transmission times are distributed as B_2. It needs investigation whether it would be beneficial to merge the two channels. The two channels are not of equal capacities and, under merging, the transmission times would become $\frac{1}{4}B_1$ for type-1 packages and $\frac{3}{4}B_2$ for type-2 packages.

It is known that $\lambda_1 = 0.2$, $\mathbb{E}B_1 = 4$, $\mathbb{V}B_1 = 1$, $\lambda_2 = 0.1$, $\mathbb{E}B_2 = 8$, $\mathbb{V}B_2 = 4$ (where $\mathbb{V}$ stands for the variance). The average *transmission delay* (i.e., the time between the moment the package is received at the channel and the moment its transmission starts) is presently 8.5 at channel 1 and 17 at channel 2.

(a) [**4pt**] What is going to be the average transmission delay at the merger channel if it transmits in the FIFO order?
Hint: Be reminded that, for a random variable X and constant a, $\mathbb{E}((aX)^2) = a^2\mathbb{E}(X^2)$.

Solution This is an $M/G/1$ system with arrival rate and service time

$$\lambda = \lambda_1 + \lambda_2 = 0.3, \quad B = \begin{cases} \frac{1}{4}B_1, & \text{wp } \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{2}{3}, \\ \frac{3}{4}B_2, & \text{wp } \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{1}{3}. \end{cases}$$

Since the order of service is FIFO, the Pollaczek-Khinchine formula applies,

$$\mathbb{E}W = \frac{\rho}{1-\rho} \cdot \frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)}.$$

We have (for $\mathbb{E}(B^2)$, we use the hint)

$$\mathbb{E}(B) = \frac{2}{3} * \mathbb{E}(\tfrac{1}{4}B_1) + \frac{1}{3} * \mathbb{E}(\tfrac{3}{4}B_2) = \frac{1}{6}\mathbb{E}B_1 + \frac{1}{4}\mathbb{E}B_2 = \frac{1}{6} \cdot 4 + \frac{1}{4} \cdot 8 = \frac{8}{3},$$

$$\mathbb{E}(B^2) = \frac{2}{3} * \mathbb{E}(\tfrac{1}{4}B_1)^2 + \frac{1}{3} * \mathbb{E}(\tfrac{3}{4}B_2)^2 = \frac{2}{3}(\tfrac{1}{4})^2 * \underbrace{\mathbb{E}(B_1^2)}_{=\mathbb{V}+(\mathbb{E})^2=1+4^2=17} + \frac{1}{3}(\tfrac{3}{4})^2 * \underbrace{\mathbb{E}(B_2^2)}_{=\mathbb{V}+(\mathbb{E})^2=4+8^2=68} \approx 13.458$$

$$\rho = \lambda\mathbb{E}(B) = 0.3 * \frac{8}{3} = 0.8,$$

and hence

$$\mathbb{E}W \approx \frac{0.8}{0.2} \cdot \frac{13.458}{2 * 8/3} \approx 10.094.$$

(b) [**4pt**] Would the merging be beneficial for the transmission times, for the transmission delays, and for the total time a package spends at the transmission channel?

Solution The transmission times are shorter after merging: $\frac{1}{4}B_1 < B_1$, $\frac{3}{4}B_2 < B_2$. So the merging is *beneficial for the transmission times* of both types of packages.

The average transmission delays (waiting times) of type-1 packages before merging, of type-2 packages before merging, and of both types of packages after merging are

$$\mathbb{E}W_1 = 8.5, \quad \mathbb{E}W_2 = 17, \quad \mathbb{E}W \approx 10.094.$$

So, the merging is *not beneficial for the transmission delays of type-1 packages* and it is *beneficial for the transmission delays of type-2 packages.*

The average total times (sojourn times) of type-1 packages before and after merging are

$$\mathbb{E}S_1 = \mathbb{E}W_1 + \mathbb{E}B_1 = 8.5 + 4 = 12.5,$$
$$\mathbb{E}S_{1,\text{merge}} = \mathbb{E}W + \tfrac{1}{4}\mathbb{E}B_1 \approx 10.094 + \tfrac{1}{4} \cdot 4 = 11.094,$$

and the average total times (sojourn times) of type-2 packages before and after merging are

$$\mathbb{E}S_2 = \mathbb{E}W_2 + \mathbb{E}B_2 = 17 + 8 = 25,$$
$$\mathbb{E}S_{2,\text{merge}} = \mathbb{E}W + \tfrac{3}{4}\mathbb{E}B_2 \approx 10.094 + \tfrac{3}{4} \cdot 8 = 16.094.$$

So, the merging is *beneficial for the total times at the channel* for both types of packages.

(c) [**2pt**] Would your conclusions from (b) still hold if, instead of transmission in the FIFO order, the next message to transmit is selected at random from the queue?

Solution Yes the conclusions from (b) still hold:

- the transmission times are not affected by the new discipline at all,
- the *average* transmission delays are still the same because the alternative discipline is work-conserving, non-size-based, and non-preemtive and hence the PK formula still applies;
- the *average* total times at the channel are still the same due to the previous two bullets.

(d) [**3pt**] For channel 1 before merging, find the expected stretch of time during which the channel is continuously transmitting packages (between two periods of having no packages to transmit).

Solution Channel 1 before merging is an $M/G/1$-FIFO system and the question is what is its mean busy period, we denote it by $\mathbb{E}(BP_1)$. From the formula sheet,

$$\mathbb{E}(BP_1) = \frac{\mathbb{E}B_1}{1 - \underbrace{\rho_1}_{=\lambda_1 \mathbb{E}B_1}} = \frac{4}{1 - 0.2 * 4} = 20.$$

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho} (1 + c_B^2) E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c / c!}{(1-\rho) \sum_{i=0}^{c-1} (c\rho)^i / i! + (c\rho)^c / c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c / c!}{\sum_{i=0}^{c} a^i / i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$