Midterm exam Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

October 25, 2023, 12:15–14:30

This exam consists of four exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. The use of books or a graphical calculator is not allowed. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. [4pt] Let X_n, $n = 0, 1, 2, \ldots$, be a discrete-time Markov chain on a state space S with transition probabilities p_{ij}, $i, j \in S$. Prove that, for all states $i_0, i_1, i_2, i_3 \in S$,

$$P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} p_{i_2 i_3}.$$

Hint: Why is it the case that

$$\begin{aligned} &P(X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &= P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0)? \end{aligned}$$

Use this fact as an inspiration for your proof.

Question 2. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(a) [6pt] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether a limit/occupancy distribution exists and find the limit/occupancy distribution in case it exists.

(b) [3pt] What is the expected number of steps it takes to reach state 1 from state 4?

(c) [2pt] What is the probability that it takes *at most* four (≤ 4) steps to reach (for the first time) state 1 from state 5?

(d) [3pt] What is the probability that it takes *at most* twenty three (≤ 23) steps to reach (for the first time) state 1 from state 5? An analytic-form answer suffices. If you use a matrix power in your answer, fully specify the matrix.

Question 3. A production system consists of two machines, machine 1 and machine 2, each of which remains in operation unless undergoing maintenance. The two machines require maintenance independently of each other. At the end of each day of production, the machine(s) in use is (are) subject to inspection, which leads to maintenance with probability $p_1 = 1/4$ for machine 1 and probability $p_2 = 1/3$ for machine 2. All necessary maintenance, for a single machine or for both machines together, takes exactly one day.

(a) [5pt] Model this production system as a discrete-time Markov chain.

(b) [4pt] What is the long-run fraction of days on which there is no production (due to simultaneous maintenance of both machines)?

(c) [3pt] Either of the machines produces a Poisson(λ) amount of products during a day, resulting in revenue r per product. Maintenance of the machines incurs costs c per machine per day. What income does this production system generate on average per day over a long time-run?

(d) [3pt] Consider a new situation where maintenance is no longer provided to a single machine, only to both machines together. When one machine requires maintenance ahead of the other, it is taken out of production until the other one requires maintenance as well. This situation can be modelled as a discrete-time Markov chain as well. Provide the adjusted transition diagram for this new situation.

Question 4. A service desk serves two types of customers, A and B, which arrive according to independent Poisson processes with rates $\lambda_A = 3$ per hour and $\lambda_B = 1$ per hour.

(a) [**2pt**] What is the probability the first customer of type A arrives before the first customer of type B?

(b) [**5pt**] What is the probability that the second customer of type A arrives before the second customer of type B?

(c) [**5pt**] Assume additionally that each customer, of type A or B, independently of the others, will require a follow-up service with probability p. What is the joint probability that the following happens: during the first hour there are at most 2 arrivals that will require a follow-up service and during the last quarter of that hour there is no arrivals at all (also no arrivals that do not require a follow-up service)?

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