SOLUTIONS

Midterm exam Stochastic Modelling

October 25, 2023

Question 1. [**4pt**] Let X_n, $n = 0, 1, 2, \ldots$, be a discrete-time Markov chain on a state space S with transition probabilities p_{ij}, $i, j \in S$. Prove that, for all states $i_0, i_1, i_2, i_3 \in S$,

$$P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} p_{i_2 i_3}.$$

Hint: Why is it the case that

$$\begin{aligned} &P(X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &= P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0)? \end{aligned}$$

Use this fact as an inspiration for your proof.

Solution We have

$$\begin{aligned} &P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &\overset{(1)}{=} P(X_3 = i_3 \mid X_2 = i_2, X_1 = i_1, X_0 = i_0) P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0) \\ &\overset{(2)}{=} P(X_3 = i_1 \mid X_2 = i_2) P(X_2 = i_1 \mid X_1 = i_1) P(X_1 = i_1 \mid X_0 = i_0) \\ &= p_{i_2 i_3} p_{i_1 i_2} p_{i_0 i_1}, \end{aligned}$$

where (2) is by the Markov property and (1) follows from the definition of conditional probability.

In more detail, (1) is true because

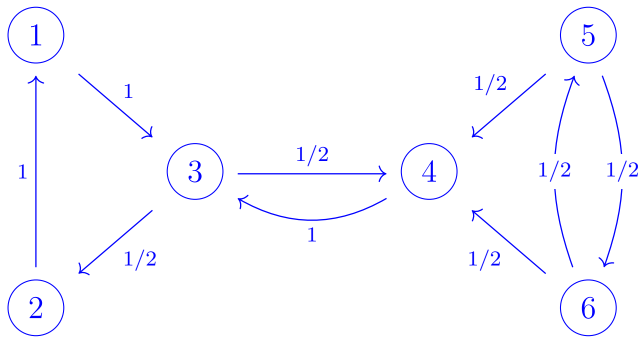
$$\begin{aligned} &P(X_3 = i_3 \mid X_2 = i_2, X_1 = i_1, X_0 = i_0) P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0) \\ &= \frac{P(X_3 = i_3, X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_2 = i_2, X_1 = i_1, X_0 = i_0)} \times \frac{P(X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_1 = i_1, X_0 = i_0)} \times \frac{P(X_1 = i_1, X_0 = i_0)}{P(X_0 = i_0)} \\ &= \frac{P(X_3 = i_3, X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_0 = i_0)} = P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \end{aligned}$$

Question 2. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(a) [6pt] For each initial state $X_0 = i$, $i = 1, 2, \dots, 6$, determine whether a limit/occupancy distribution exists and find the limit/occupancy distribution in case it exists.

Solution Looking at the transition diagram,



there are two communicating classes:

- {1,2,3,4} is absorbing,
- {5,6} is non-absorbing.

Consider the absorbing class $\{1, 2, 3, 4\}$ in isolation. It is finite, hence it has an occupancy distribution. It is also aperiodic (e.g., it is possible to get from 1 back to 1 in 3 steps: via 3,2, and in 5 steps: via 3,4,3,2). Hence, it also has a limit distribution. To find both the occupancy and limit distribution of {1,2,3,4} in isolation, we solve the system of balance and normalization equations for this class:

$$\begin{cases} \pi_1 = \pi_2, \\ \pi_2 = \pi_3 * 1/2, \\ \pi_3 = \pi_1 + \pi_4, \\ \pi_4 = \pi_3 * 1/2, \quad \text{1st, 2nd, 3rd lines give } \pi_1 = \pi_2, \pi_3 = 2\pi_2, \pi_4 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1. \quad (1+1+2+1)\pi_2 = 1 \Rightarrow \pi_2 = 1/5 = \pi_1 = \pi_4, \pi_3 = 2/5 \end{cases}$$

If the MC starts in the non-absorbing class {5,6}, it will end up in the absorbing class eventually with its limit and occupancy distribution. Hence, for any initial state $X_0 = 1, 2, 3, 4, 5, 6$, the occupancy and limit distribution exist and are both given by

$$\pi^{occ} = \pi^{lim} = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, 0, 0).$$

(b) [**3pt**] What is the expected number of steps it takes to reach state 1 from state 4?

Solution Let $T_1 := \min\{n \geq 0\colon X_n = 1\}$ and $m_i := E(T_1 \mid X_0 = i)$. The question is what is m_4. By conditioning on the 1st step we get the system

$$\begin{cases} m_2 = 1, \\ m_3 = 1 + 1/2m_2 + 1/2m_4, \quad 1)\ m_3 = 3/2 + 1/2m_4, \\ m_4 = 1 + m_3 \quad 2)\ m_4 = 5/2 + 1/2m_4 \Rightarrow 1/2m_4 = 5/2 \Rightarrow m_4 = 5. \end{cases}$$

(c) [**2pt**] What is the probability that it takes *at most* four (≤ 4) steps to reach (for the first time) state 1 from state 5?

Solution

$$P(T_1 \leq 4 | X_0 = 5) = p_{54}p_{43}p_{32}p_{21} = \tfrac{1}{2} \cdot 1 \cdot \tfrac{1}{2} \cdot 1 = \tfrac{1}{4}.$$

(d) [**3pt**] What is the probability that it takes *at most* twenty three (≤ 23) steps to reach (for the first time) state 1 from state 5? An analytic-form answer suffices. If you use a matrix power in your answer, fully specify the matrix.

Solution

$$P(T_1 \leq 23 | X_0 = 5) = (\tilde{P}^{23})_{51},$$

where

$$\tilde{P} = \begin{pmatrix} \mathbf{1} & 0 & \mathbf{0} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

i.e. $\tilde{P}$ is the transition matrix of the adjusted Markov Chain with state 1 made absorbing (the red transition probabilities are different than in P, the rest are the same as in P).

Question 3. A production system consists of two machines, machine 1 and machine 2, each of which remains in operation unless undergoing maintenance. The two machines require maintenance independently of each other. At the end of each day of production, the machine(s) in use is (are) subject to inspection, which leads to maintenance with probability $p_1 = 1/4$ for machine 1 and probability $p_2 = 1/3$ for machine 2. All necessary maintenance, for a single machine or for both machines together, takes exactly one day.

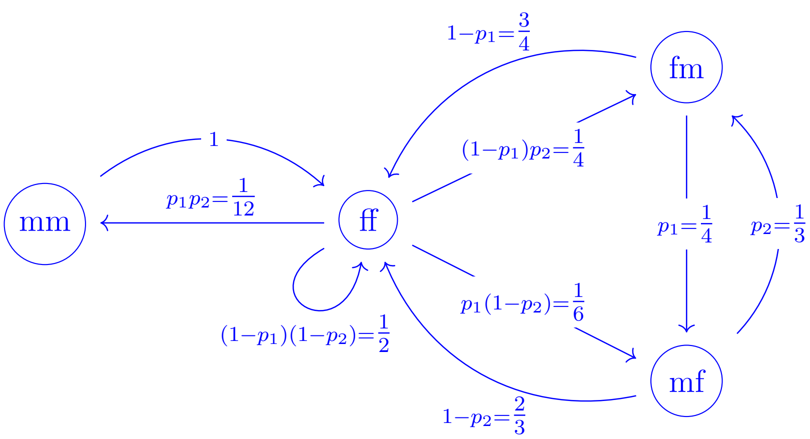
(a) [5pt] Model this production system as a discrete-time Markov chain.

Solution Let

$$X_n = (\text{state of machine 1, state of machine 2}) \text{ on day } n,$$

where each of the machines can have the following states:
- "f" if it is functioning
- or "m" if it is in maintenance.

$\{X_n\}_{n\geq 0}$ is a Markov chains with the transition diagram



The *Markov property* follows implicitly from the formulation of the exercise: only today's state of the machine matters for tomorrow's state of the machine, any earlier states of the machine do not matter. The *time-homogeneity* is in place also, and that is because the probabilities of going into maintenance do not depend on day n.

(b) [4pt] What is the long-run fraction of days on which there is no production (due to simultaneous maintenance of both machines)?

Solution The answer to the question is π^{occ}_{mm}. Since the MC is finite and irreducible, the system of balance and normalization equations gives π^{occ}. We have

$$\begin{cases} \pi_{mm} = \pi_{ff}\frac{1}{12}, \\ \pi_{ff}\frac{1}{2} = \pi_{mm} + \pi_{fm}\frac{3}{4} + \pi_{mf}\frac{2}{3}, \\ \pi_{fm} = \pi_{ff}\frac{1}{4} + \pi_{mf}\frac{1}{3}, \quad 1)\pi_{fm} = \pi_{ff}\frac{1}{4} + \pi_{ff}\frac{1}{18} + \pi_{fm}\frac{1}{12} \Rightarrow \frac{11}{12}\pi_{fm} = \frac{11}{36}\pi_{ff} \Rightarrow \pi_{fm} = \frac{1}{3}\pi_{ff} \\ \pi_{mf} = \pi_{ff}\frac{1}{6} + \pi_{fm}\frac{1}{4}, \quad 2)\pi_{mf} = \pi_{ff}\frac{1}{6} + \pi_{ff}\frac{1}{12} = \frac{1}{4}\pi_{ff} \\ \pi_{mm} + \pi_{ff} + \pi_{fm} + \pi_{mf} = 1 \quad 3)(\frac{1}{12} + 1 + \frac{1}{3} + \frac{1}{4})\pi_{ff} = \frac{20}{12}\pi_{ff} = \frac{5}{3}\pi_{ff} = 1 \Rightarrow \pi_{ff} = \frac{3}{5} \end{cases}$$

i.e.

$$\pi^{occ} := (\pi_{mm}, \pi_{ff}, \pi_{fm}, \pi_{mf})^{occ} = (\tfrac{1}{20}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{20}).$$

The final answer is $\pi^{occ}_{mm} = 1/20$.

(c) [3pt] Either of the machines produces a Poisson(λ) amount of products during a day, resulting in revenue r per product. Maintenance of the machines incurs costs c per machine per day. What income does this production system generate on average per day over a long time-run?

Solution The generic revenue in each state is:

$$R_{mm} = 0, \quad R_{fm}, R_{mf} \sim r{\cdot}Poi(\lambda), \quad R_{ff} \sim r{\cdot}(Poi_1(\lambda){+}Poi_2(\lambda)) \sim r{\cdot}Poi(2\lambda).$$

The generic maintenance costs in each state are :

$$C_{mm} = 2c, \quad C_{fm} = C_{mf} = c, \quad C_{ff} = 0.$$

The income is the revenue minus the costs. Hence, the income on average per day over a long time-run is

$$\begin{aligned}\big(\pi^{occ}_{mm}ER_{mm} + \pi^{occ}_{ff}ER_{ff} + \pi^{occ}_{fm}ER_{fm} + \pi^{occ}_{mf}ER_{mf}\big) -\\ \big(\pi^{occ}_{mm}EC_{mm} + \pi^{occ}_{ff}EC_{ff} + \pi^{occ}_{fm}EC_{fm} + \pi^{occ}_{mf}EC_{mf}\big)\\ = \big(\tfrac{3}{5}\cdot 2r\lambda + (\tfrac{1}{5}+\tfrac{3}{20})\cdot r\lambda\big) - \big(\tfrac{1}{20}\cdot 2c + (\tfrac{1}{5}+\tfrac{3}{20})\cdot c\big)\\ = \frac{12\cdot 2+4+3}{20}\cdot r\lambda - \frac{2+4+3}{20}\cdot c = \tfrac{31}{20}r\lambda - \tfrac{9}{20}c = 1.55r\lambda - 0.45c.\end{aligned}$$

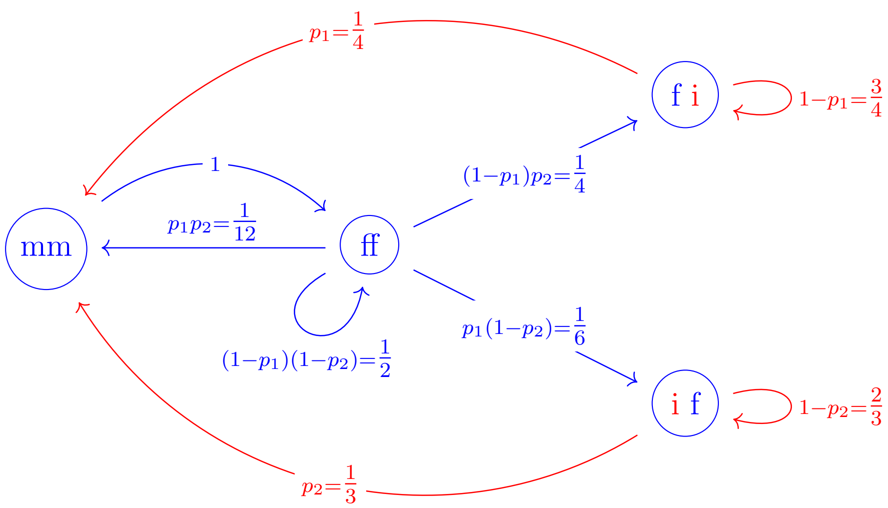
(d) [**3pt**] Consider a new situation where maintenance is no longer provided to a single machine, only to both machines together. When one machine requires maintenance ahead of the other, it is taken out of production until the other one requires maintenance as well. This situation can be modelled as a discrete-time Markov chain as well. Provide the adjusted transition diagram for this new situation.

Solution We still work with the MC

$$X_n = (\text{state of machine 1, state of machine 2) on day } n,$$

but now each machine can be in the following states:
-"f" if it is functioning,
- "i" if it is idling while waiting for maintenance (i.e., waiting for the other machine to require maintenance),
- or "m" if it is in maintenance.

The transition diagram is now as follows (what is different compared to the previous situation is in red):



Question 4. A service desk serves two types of customers, A and B, which arrive according to independent Poisson processes with rates $\lambda_A = 3$ per hour and $\lambda_B = 1$ per hour.

(a) **[2pt]** What is the probability the first customer of type A arrives before the first customer of type B?

Solution Denote by A_1 (B_1) the arrival time of the first type-A (type-B) customer. The question is what is $P(A_1 < B_1)$. Since $A_1 \sim \text{Exp}(\lambda_A)$ and $B_1 \sim \text{Exp}(\lambda_B)$ are independent, we have

$$P(A_1 < B_1) = P(A_1 \text{ wins from } B_1) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{3}{4}.$$

(b) **[5pt]** What is the probability that the second customer of type A arrives before the second customer of type B?

Solution Denote Denote by A_2 (B_2) the time between the 1st and 2nd arrivals of type A (type B). Also we introduce a notation for the event of interest,

$$E := \{\text{2nd type } A \text{ customer arrives before 2nd type } B \text{ customer}\}.$$

There are three possible scenarios for the event E to happen:

- $E_1 := \{A_1 \text{ wins from } B_1,\ A_2 \text{ wins from remaining } B_1\}$,
- $E_2 := \{A_1 \text{ wins from } B_1,\ A_2 \text{ loses to remaining } B_1,\ \text{remaining } A_2 \text{ wins from } B_2\}$,
- $E_3 := \{A_1 \text{ loses to } B_1,\ \text{remaining } A_1 \text{ wins from } B_2,\ A_2 \text{ wins from remaining } B_2\}$.

The inter-arrival times A_i are $\text{Exp}(\lambda_A)$ and due to the memoryless property, the remaining inter-arrival times A_i are $\text{Exp}(\lambda_A)$ as well. Similarly, the inter-arrival and remaining inter-arrival times B_i are $\text{Exp}(\lambda_B)$. Also there is independence between the pairs of competing exponentials in E_1, E_2, E_3. Hence, we have

$$\begin{aligned} P(E) &= P(E_1) + P(E_2) + P(E_3) \\ &= \frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} + \frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_B}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} + \frac{\lambda_B}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} \\ &= \frac{3}{4}\frac{3}{4} + \frac{3}{4}\frac{1}{4}\frac{3}{4} + \frac{1}{4}\frac{3}{4}\frac{3}{4} = \frac{27}{32}. \end{aligned}$$

(c) [5pt] Assume additionally that each customer, of type A or B, independently of the others, will require a follow-up service with probability p. What is the joint probability that the following happens: during the first hour there are at most 2 arrivals that will require a follow-up service and during the last quarter of that hour there is no arrivals at all (also no arrivals that do not require a follow-up service)?

Solution Let
- $N_A(\cdot)$ and $N_B(\cdot)$ denote the arrival processes of, respectively, type A and type B customers;
- $N(\cdot) := N_A(\cdot) + N_B(\cdot)$ be the total arrival process of all customers; note $N(\cdot) \sim PP(\lambda_A + \lambda_B) = PP(4)$ by merging;
- $N_{\text{follow-up}}(\cdot)$ denote the arrival process of customers of type A and type B who require a follow-up service; note $N_{\text{follow-up}}(\cdot) \sim PP((\lambda_A + \lambda)p) = PP(4p)$ by thinning.

The question is what is

$$P\big(N_{\text{follow-up}}(1) \le 2, N(\tfrac{3}{4}, 1] = 0\big) = P\big(N_{\text{follow-up}}(\tfrac{3}{4}) \le 2, N(\tfrac{3}{4}, 1] = 0\big)$$
$$= P\big(N_{\text{follow-up}}(\tfrac{3}{4}) \le 2\big) \cdot P(N(\tfrac{3}{4}, 1] = 0) \quad \text{independence on non-overlapping intervals}$$
$$= P\big(Poi(4p \cdot \tfrac{3}{4}) \le 2\big) \cdot P\big(Poi(4 \cdot \tfrac{1}{4}) = 0\big)$$
$$= e^{-3p}(1 + 3p + (3p)^2/2) \cdot e^{-1} = e^{-3p-1}(1 + 3p + (3p)^2/2).$$