Resit Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

February 13, 2023, 18:45–21:30

This exam consists of five exercises, for which you can obtain 54 points in total. Your grade will be calculated as (number of points + 6)/6. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) [5pt] (i) Is it correct, that a limit distribution exists for all initial states $X_0 = i$, $i = 1, 2, \ldots, 6$? (ii) For which of the initial states $X_0 = i$, $i = 1, 2, \ldots, 6$, does an occupancy distribution exist? Find the occupancy distribution if it exists.

(b) [3pt] Assume the initial state is 1. Find the probability that the Markov chain will ever reach state 6.

Question 2. An employee handles a certain type of tasks which come up at most once per day. For efficiency reasons, the employee prefers to handle such tasks in batches of two but sometimes he handles just a single task. To be precise, on any given day, independently of the previous days, a single task comes up with probability 1/2 or no task comes up with probability

1/2. A newly generated task is never handled on the same day, it will be handled the next day or later. If, at the end of a day, there is a single open task, the employee handles it the next day with probability 1/5. Otherwise, the employee will wait till a second task comes up and then handle the two tasks together the day after. Each decision on whether to wait for a second task is independent from previous such decisions.

(a) **[5pt]** Argue that the numbers of open tasks at the end of each day form a discrete-time Markov chain.

(b) **[4pt]** Calculate the fraction of days that end with exactly one open task.

(c) **[3pt]** An employee completes a single task in 2 hours and a batch of two tasks in 3 hours. Calculate the amount of time the employee spends on average per day on this type of tasks.

Question 3. A small supermarket has two checkouts. Customers arrive to the checkout area according to a Poisson process with rate λ per hour. Regardless of the queue sizes in front of the two checkouts, each customer chooses the 1st checkout with probability 2/3 (and the 2nd checkout with probability 1/3).

(a) **[5pt]** Determine the joint probability that at least two customers arrive to the first checkout between 9:00 and 11:00 but no customers at all arrive to the checkout area between 9:30 and 10:00.

The service times at the 1st checkout follow an exponential distribution with rate 12 per hour; the service times and the 2nd checkout follow an exponential distribution with rate 8 per hour. When you and your friend arrive to the checkout area, two customers are already being served, one at each checkout. You queue up for the 1st checkout, your friend for the 2nd.

(b) **[5pt]** (i) What is the probability that you will *start* checking out before your friend? (ii) What is the probability that you will *finish* checking out before your friend?

Question 4. In addition to its standard ambulances, a region wants to invest into specialised ambulances called Mobile Intensive Care Units (MICU), which are better suited for transportation of patients in a critical condition. Data show that such transportations requests (IC transportation requests) arrive according to a Poisson process with an average of 16 patients per day and the total duration to transport a patient approximately follows an exponential distribution with an average of 2 hours.

For starters, the region has bought one MICU. IC transportation requests get assigned to this MICU unless it is occupied and there are already 2 other patients waiting for it. (IC transportation requests that are rejected by the MICU, are carried out by the standard ambulances.)

(a) [**4pt**] Argue that the number of IC transportation requests assigned to the MICU is a continuous-time Markov chain.

(b) [**5pt**] (i) Find the fraction of time the MICU is idling. (ii) Find the fraction of IC transportation requests rejected by the MICU.

(c) [**4pt**] Give the Little's law for the requests that (got assigned to the MICU) but have to wait for the MICU. Use the Little's law to calculate the average waiting time among those who get assigned to the MICU.

Question 5. Customers arrive to a single-server system according to a Poisson process of rate λ; they are served in the order of arrival. The service of each customer consists of two consecutive stages, both distributed exponentially, the rate is μ_1 for the 1st stage and μ_2 for the 2nd stage. You can assume independence between the two stages of a service time.

(a) [**2pt**] Under what condition on λ, μ_1, and μ_2 is this system stable?

(b) [**4pt**] For $\lambda = 1$ and $\mu_1 = \mu_2 = 4$, find the customer-average waiting time.
Reminder: For a random variable $X \sim \text{Exponential}(\alpha)$, the variance is $1/\alpha^2$. For independent random variables, the variance of the sum is the sum of the variances.

(c) [**5pt**] Assume arbitrary λ, μ_1, and μ_2 rather than the values from (b). Model the system as a continuous-time Markov chain.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$
$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$
$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$
$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$