SOLUTIONS

Resit Stochastic Modelling

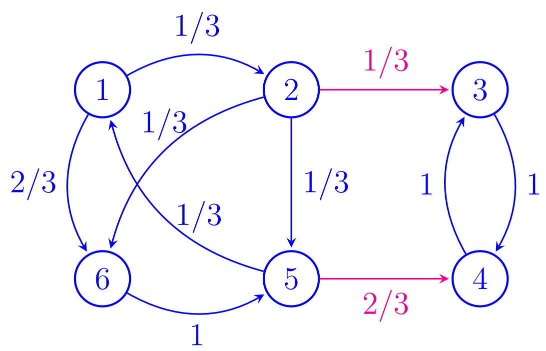
February 13, 2023

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) [**5pt**] (i) Is it correct, that a limit distribution exists for all initial states $X_0 = i$, $i = 1, 2, \ldots, 6$? (ii) For which of the initial states $X_0 = i$, $i = 1, 2, \ldots, 6$, does an occupancy distribution exist? Find the occupancy distribution if it exists.

Solution Looking at the transition diagram



there are two communicating classes:

- {1,2,5,6} is transient,
- {3,4} is absorbing, has period 2.

(i) No, the statement is incorrect. Due to the periodicty, no limit distribution exists e.g. for $X_0 = 3$.

(ii) For any initial state, the MC ends up in the absorbing class $\{3, 4\}$, which has (1/2, 1/2) as its occupancy distribution, for symmetry reasons. I.e., for any initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, the occupancy distribution exists and is given by

$$\pi^{occ} = (0, 0, 1/2, 1/2, 0, 0).$$

(b) [**3pt**] Assume the initial state is 1. Find the probability that the Markov chain will ever reach state 6.

Solution The question is what is q_1, where

$$q_i := P(\text{ever reach } 6 \mid X_0 = i).$$

By conditioning on the 1st step we get the system

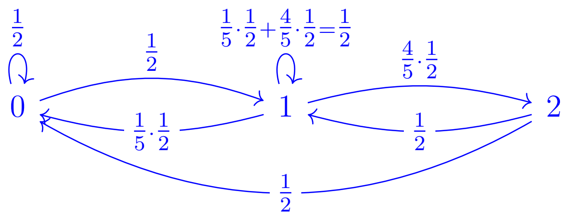
$$\begin{cases} q_1 = \frac{1}{3}q_2 + \frac{2}{3} * 1, \overset{\text{plug in } q_2}{=} \frac{1}{3}(\frac{1}{3}q_5 + \frac{1}{3}) + \frac{2}{3} \overset{\text{plug in } q_5}{=} \frac{1}{3}(\frac{1}{9}q_1 + \frac{1}{3}) + \frac{2}{3} = \frac{1}{27}q_1 + \frac{7}{9} \\ q_2 = \frac{1}{3} * 0 + \frac{1}{3}q_5 + \frac{1}{3} * 1, \\ q_5 = \frac{1}{3}q_1 + \frac{2}{3} * 0. \end{cases}$$

Finally, $q_1 = \frac{1}{27}q_1 + \frac{7}{9}$ gives $q_1 = 21/26$.

Question 2. An employee handles a certain type of tasks which come up at most once per day. For efficiency reasons, the employee prefers to handle such tasks in batches of two but sometimes he handles just a single task. To be precise, on any given day, independently of the previous days, a single task comes up with probability 1/2 or no task comes up with probability 1/2. A newly generated task is never handled on the same day, it will be handled the next day or later. If, at the end of a day, there is a single open task, the employee handles it the next day with probability 1/5. Otherwise, the employee will wait till a second task comes up and then handle the two tasks together the day after. Each decision on whether to wait for a second task is independent from previous such decisions.

(a) [**5pt**] Argue that the numbers of open tasks at the end of each day form a discrete-time Markov chain.

Solution Let by X_n be the number of open tasks at the end of day n, $n \geq 0$. This is a DTMC with transition diagram



To explain some of the transition probabilities,

- $p_{10} = \frac{1}{5} \cdot \frac{1}{2}$ corresponds to the employee handling the single task next day and no new task coming up next day,
- $p_{11} = \frac{1}{5} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{2} = \frac{1}{2}$ corresponds to the employee handling the single task next day and a new task coming up next day OR the employee not handling the single task next day and no new task coming up next day,
- $p_{12} = \frac{4}{5} \cdot \frac{1}{2}$ corresponds to the employee not handling the single task next day and a new task coming next day,
- note that p_{11} can also be calculated as $p_{11} = 1 - p_{10} - p_{12} = \frac{1}{2}$,
- as for transitions out of 2, the employee will for sure handle the 2 tasks next day and hence it only matters whether the new task comes up next day.

The *Markov property* follows from the independence of the task arrivals and the employee's decisions about the single tasks. We also have the *time-homogeneity* as the transition probabilities do not depend on time n.

(b) [**4pt**] Calculate the fraction of days that end with exactly one open task.

Solution The question is what is π_1^{occ}.

Since this MC is irreducible and has a finite state space, it has a π^{occ}, which is determined by the system

$$\begin{cases} \text{balance} & \pi_0 \frac{1}{2} = \pi_1 \frac{1}{2} \cdot \frac{1}{5} + \pi_2 \frac{1}{2}, \\ & \pi_1 \frac{1}{2} = \pi_0 \frac{1}{2} + \pi_2 \frac{1}{2}, \quad 2)\ \pi_1 = \pi_0 + \pi_2 \Rightarrow \pi_0 = \pi_1 - \pi_2 = \pi_1 \frac{3}{5} \\ & \pi_2 = \pi_1 \frac{1}{2} \cdot \frac{4}{5}, \quad 1)\ \pi_2 = \pi_1 \frac{2}{5} \\ \text{norm} & \pi_0 + \pi_1 + \pi_2 = 1 \quad 3)\ (\frac{3}{5} + 1 + \frac{2}{5})\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{2}. \end{cases}$$

Hence, the final answer to (b) is $\pi_1^{occ} = \frac{1}{2}$.

Also, as will be relevant for (c), the full occupancy distribution is

$$\pi^{occ} = (\tfrac{3}{10}, \tfrac{1}{2}, \tfrac{1}{5}).$$

(c) [3pt] An employee completes a single task in 2 hours and a batch of two tasks in 3 hours. Calculate the amount of time the employee spends on average per day on this type of tasks.

Solution We will view the hours the employee will be spending the next day as costs. The generic costs in each state are:

$$C_0 = 0, \quad C_2 = 3,$$
$$C_1 = \begin{cases} 2 & \text{wp } 1/5, \quad \text{if decides to work on the single task next day} \\ 0 & \text{wp } 4/5, \quad \text{otherwise} \end{cases}$$

Hence the long-run average costs per day, or time spend per day, is

$$\pi_0^{occ} EC_0 + \pi_1^{occ} EC_1 + \pi_2^{occ} EC_2 = 0 + \tfrac{1}{2} \cdot (2 \cdot \tfrac{1}{5}) + \tfrac{1}{5} \cdot 3 = \tfrac{4}{5} \text{ hr} = 48 \text{ min}.$$

Question 3. A small supermarket has two checkouts. Customers arrive to the checkout area according to a Poisson process with rate λ per hour. Regardless of the queue sizes in front of the two checkouts, each customer chooses the 1st checkout with probability 2/3 (and the 2nd checkout with probability 1/3).

(a) [5pt] Determine the joint probability that at least two customers arrive to the first checkout between 9:00 and 11:00 but no customers at all arrive to the checkout area between 9:30 and 10:00.

Solution Let
- $N(t), t \geq 0$ denote the total arrival process to the checkout area,
- $N_1(t), t \geq 0$ denote the arrival process to the 1st checkout. By *thinning*, this is a Poisson process, too, with rate $\frac{2}{3}\lambda$.

The question is what is

$$\begin{aligned}
&P(N_1(9,11] \geq 2, N(9\tfrac{1}{2},10] = 0)\\
&= P(N_1(9,9\tfrac{1}{2}] + N_1(10,11] \geq 2, N(9\tfrac{1}{2},10] = 0) \quad \text{ind. on disjoint intervals}\\
&= P(N_1(9,9\tfrac{1}{2}] + N_1(10,11] \geq 2) * P(N(9\tfrac{1}{2},10] = 0) \quad \text{merge ind. Poi. r.v.'s in the 1st probabi}\\
&= P(\text{Poi}(\underbrace{\tfrac{2}{3}\lambda \cdot \tfrac{1}{2} + \tfrac{2}{3}\lambda \cdot 1}_{=\lambda}) \geq 2) * P(\text{Poi}(\lambda \cdot \tfrac{1}{2}) = 0)\\
&= \big(1 - P(\text{Poi}(\lambda) \geq 1)\big) * P(\text{Poi}(\lambda \cdot \tfrac{1}{2}) = 0) = \big(1 - e^{-\lambda}(1+\lambda)\big) * e^{-\lambda/2}.
\end{aligned}$$

The service times at the 1st checkout follow an exponential distribution with rate 12 per hour; the service times and the 2nd checkout follow an exponential distribution with rate 8 per hour. When you and your friend arrive to the checkout area, two customers are already being served, one at each checkout. You queue up for the 1st checkout, your friend for the 2nd.

(b) [5pt] (i) What is the probability that you will *start* checking out before your friend? (ii) What is the probability that you will *finish* checking out before your friend?

Solution We refer to the customer served at checkout 1 as A, and the customer served at checkout 2 as B, and the friend as F. I.e., when me and the friend arrive, the queues look as follows:

1, rate 12:	A	2, rate 8:	B
	I		F

In (i), the answer is

$$P(\underbrace{\text{A}}_{\sim Exp(12)} \text{ wins from } \underbrace{\text{B}}_{\sim Exp(8)}) = \frac{12}{12+8} = \frac{3}{5}.$$

(ii) happens in the following three scenarios (they are disjoint, that is why we will add up their probabilities):
- A wins from B, then I win from *(the remaining)* B,
- A wins from B, then I loose to *(the remaining)* B, then *(the remaining)* I win from F,
- A looses to B, then *(the remaining)* A wins from F, then I win from *(the remaining)* F.

Using the *memoryless property* of the exponential distribution, the answer to (ii) can be computed as

$$
\begin{aligned}
&P(\text{Exp}(12) \text{ wins Exp}(8)) \cdot P(\text{Exp}(12) \text{ wins Exp}(8)) \\
&+ P(\text{Exp}(12) \text{ wins Exp}(8)) \cdot P(\text{Exp}(12) \text{ looses Exp}(8)) \cdot P(\text{Exp}(12) \text{ wins Exp}(8)) \\
&+ P(\text{Exp}(12) \text{ looses Exp}(8)) \cdot P(\text{Exp}(12) \text{ wins Exp}(8)) \cdot P(\text{Exp}(12) \text{ wins Exp}(8)) \\
&= \frac{3}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} = \frac{81}{125}.
\end{aligned}
$$

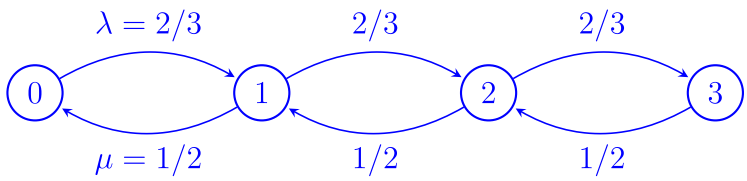
Question 4. In addition to its standard ambulances, a region wants to invest into specialised ambulances called Mobile Intensive Care Units (MICU), which are better suited for transportation of patients in a critical condition. Data show that such transportations requests (IC transportation requests) arrive according to a Poisson process with an average of 16 patients per day and the total duration to transport a patient approximately follows an exponential distribution with an average of 2 hours.

For starters, the region has bought one MICU. IC transportation requests get assigned to this MICU unless it is occupied and there are already 2 other patients waiting for it. (IC transportation requests that are rejected by the MICU, are carried out by the standard ambulances.)

(a) **[4pt]** Argue that the number of IC transportation requests assigned to the MICU is a continuous-time Markov chain.

Solution IC transportations have
- arrival rate $\lambda = 16/24$ p/hr $= 2/3$ p/hr,
- service rate $\mu = 1/2$ p/hr.

Let $L(t)$ be the number of requests assigned to the MICU at time t, including the patient being transported and up to 2 patients waiting. It is a CTMC with transition diagram



I.e., this is an $M/M/1/3$ model.

(b) [**5pt**] (i) Find the fraction of time the MICU is idling. (ii) Find the fraction of IC transportation requests rejected by the MICU.

Solution The question is what is (i) p_0^{occ}, (ii) by *PASTA*, p_3^{occ}. Since $L(\cdot)$ is an irreducible CTMC on a finite state space, the occupancy distribution exists and solves the balance and normalization equations:

$$\begin{cases} p_0 * \frac{2}{3} = p_1 * \frac{1}{2}, & \text{balance for state } 0 \\ p_1 * \frac{2}{3} = p_2 * \frac{1}{2}, & \text{balance for set } \{0,1\} \\ p_2 * \frac{2}{3} = p_3 * \frac{1}{2}, & \text{balance for set } \{0,1,2\} \\ \sum_{i=0}^{4} p_i = 1. & \end{cases}$$

From the balance equations it follows that $p_i = (\frac{4}{3})^i p_0$ and then the normalization equation gives $p^{occ} = (\frac{27}{175}, \frac{36}{175}, \frac{48}{175}, \frac{64}{175})$.

I.e. the answers are (i) 27/175, (ii) 64/175.

(c) [**4pt**] Give the Little's law for the requests that (got assigned to the MICU) but have to wait for the MICU. Use the Little's law to calculate the average waiting time among those who get assigned to the MICU.

Solution Of all IC transportation requests, fraction $1 - p_3^{occ}$ get assgined to the MICU and hence, the Little's law is

$$EL^q = \lambda(1 - p_3^{occ})EW.$$

Then

$$EW = \frac{EL^q}{\lambda(1 - p_3^{occ})} = \frac{1 * p_2^{occ} + 2 * p_3^{occ}}{\lambda(1 - p_3^{occ})} \stackrel{\text{(b)}}{=} \frac{(48 + 2 * 64)/175}{2/3 * (175 - 64)/175} = \frac{88}{37} \text{ hr} \approx 2.38 \text{ hr} \approx 2\text{hr } 23 \text{ min}.$$

Question 5. Customers arrive to a single-server system according to a Poisson process of rate λ; they are served in the order of arrival. The service of each customer consists of two consecutive stages, both distributed exponentially, the rate is μ_1 for the 1st stage and μ_2 for the 2nd stage. You can assume independence between the two stages of a service time.

(a) [**2pt**] Under what condition on λ, μ_1, and μ_2 is this system stable?

Solution This is an $M/G/1$ system with arrival rate λ and service time $B = B_1 + B_2$, where $B_1 \sim \text{Exp}(\mu_1)$ and $B_2 \sim \text{Exp}(\mu_2)$ are independent. The stability condition is

$$\rho := \lambda \mathbb{E}B < 1, \qquad \text{i.e., } \lambda(\frac{1}{\mu_1} + \frac{1}{\mu_2}) < 1.$$

(b) [**4pt**] For $\lambda = 1$ and $\mu_1 = \mu_2 = 4$, find the customer-average waiting time.
Reminder: For a random variable $X \sim \text{Exponential}(\alpha)$, the variance is $1/\alpha^2$. For independent random variables, the variance of the sum is the sum of the variances.

Solution Since the order of service is FIFO, the Pollaczek-Khinchine formula applies,

$$\mathbb{E}W = \frac{\rho}{1-\rho} \cdot \frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)}.$$

We have

$$\begin{aligned}
\mathbb{E}(B) &= \tfrac{1}{4} + \tfrac{1}{4} = 1/2, \\
\mathbb{E}(B^2) &= \underbrace{\mathbb{V}B}_{=\mathbb{V}B_1 + \mathbb{V}B_2} + (\mathbb{E}B)^2 = \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{4} = 3/8, \\
\rho &= \lambda \mathbb{E}(B) = 1 * 1/2 = 1/2,
\end{aligned}$$

and hence

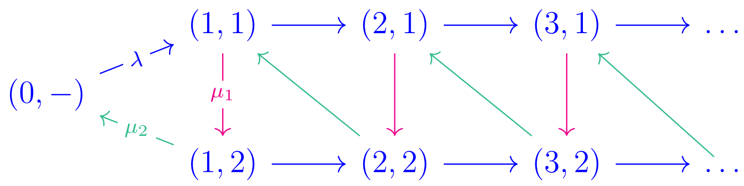
$$\mathbb{E}W = \frac{1/2}{1/2} \cdot \frac{3/8}{2 * 1/2} = 3/8.$$

(c) [**5pt**] Assume arbitrary λ, μ_1, and μ_2 rather than the values from (b). Model the system as a continuous-time Markov chain.

Solution Let

$$X(t) = (\#\ \text{customers present}, \text{stage in service}) \quad \text{at time } t.$$

This is a CTMC with transition diagram



(with all $\xrightarrow{\lambda}$, $\xrightarrow{\mu_1}$, $\xrightarrow{\mu_2}$).

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\,\mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$