Final exam Stochastic Modelling (X_400646)

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This exam consists of three exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider an $M/M/2/4$ system where customers (attempt to) arrive at rate $\lambda = 2$ and each of the two servers has service rate $\mu = 1$. Recall that, next to the two spots at the servers, this system only has two spots in the waiting room. Those customers which, upon arrival, do not find a free server or a free spot in the waiting room are lost.

(a) [2pt] Formulate a CTMC based on which you can answer the subsequent parts of the question.

(b) [5pt] What is the fraction of time that each of the following situations occur: (i) the system is empty, (ii) the waiting room is full?

(c) [2pt] What fraction of customers are lost and why?

(d) [2pt] What is the time-average number of customers in the waiting room?

(e) [4pt] Now assume that the 2 servers are not identical: one is faster and the other is slower. The service times at the fast server are distributed exponentially with rate $\mu_1 = 1.5$, and at the slow server - exponentially with rate $\mu_2 = 0.5$. A customer only has a choice which server to go to if that customer arrives into an empty system; in that case the customer goes to the fast server. In all other cases, the customer goes to whichever server is available at the moment. Model this new situation as a CTMC.
Hint: this new situation requires one state more than the original situation.

Question 2. Consider a channel that transmits packages one at a time and has an infinite buffer (i.e., an infinite queue for packages). The packages are generated according to a Poisson process of rate λ; all transmission times are independent and distributed exponentially with rate μ. There is also a *patience threshold* K, meaning: if a new package is generated when there are already K or more packages at the channel (including the package currently in transmission), then the new package remains at this channel with probability p or gets rerouted elsewhere with probability $1-p$.

(a) [4pt] Argue that the number of packages at the channel (in total in transmission and in the buffer) is a CTMC.

(b) [2pt] Argue intuitively what is the stability condition for this CTMC. In particular, does the specific value of K matter?

(c) [5pt] Find the limit and occupancy distribution. In particular, derive that

$$p_0^{lim} = p_0^{occ} = \left[\sum_{i=0}^{K-1} (\lambda/\mu)^i + \frac{(\lambda/\mu)^K}{1-\lambda p/\mu} \right]^{-1}.$$

Hint: For normalisation, group states $i \geq K$.

(d) [4pt] Derive the fraction of rerouted packages *based on your answer to (c)*. In particular, if you obtained the correct answer in (c), you should obtain the fraction of rerouted customers

$$\Pi_{loss} = (1-p)\frac{(\lambda/\mu)^K}{1-\lambda p/\mu}\, p_0^{occ}.$$

Question 3. Jobs arrive at a server according to a Poisson process of rate $\lambda = 1/3$. For $1/4$ of the jobs, their service times have a normal $N(4, 1^2)$ distribution. The remaining $3/4$ of the jobs require a fixed service time 2.

(a) [5pt] Assume the FIFO discipline at the server and find the average waiting time $\mathbb{E}W$.

(b) [3pt] The system under consideration is the result of pooling. Previously, the varying-size jobs and fixed-size jobs were served at two separate servers, each twice as slow as the server in the present system. Has the pooling been beneficial for the fixed-sized jobs? Why yes or why not?

For questions (c) and (d) assume that, in the pooled system, the service discipline is not FIFO but instead the $N(4, 1^2)$-sized jobs have a non-preemptive priority over the jobs of fixed size 2.

(c) [2pt] Is the Pollaczek-Khinchine formula applicable in this new situation in order to find the average waiting time across all jobs together? Why yes or why not?

(d) [5pt] Determine the average waiting time $\mathbb{E}W_1$ of the high-priority jobs by doing Mean Value Analysis for this group.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho} (1 + c_B^2) E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1} (c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$