YOUR NAME:
YOUR TA usually:

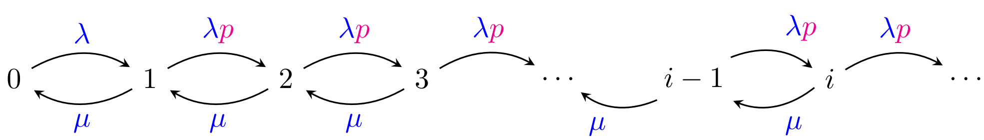
SOLUTIONS Stochastic Modelling, Short test 3

21 November 2022, 12:25-12:45

Question 1. Consider the following modification of the $M/M/1$ queue with arrival rate λ and service rate μ: now the arriving customers that would have to wait to get service join the system with probability p (and with probability $1-p$, they do not join the system because they do not wish to wait). That is, now λ is the rate of *arrival attempts*.

(a) $L(t) :=$ number of customers in this system at time t, $t \geq 0$, is a CTMC. Fill in the rates in the transition diagram of this CTMC:

Solution



(b) Intuitively, under which condition for λ, μ, p is this CTMC stable?

Solution $\lambda p < \mu$

(c) Write down the balance equations for sets $\{0, \ldots, i-1\}$, $i \geq 1$.

Solution

$$p_0 * \lambda = p_1 * \mu,$$
$$\text{for } i \geq 2, \quad p_{i-1} * \lambda p = p_i * \mu.$$

(d) From the balance equations, express in terms of p_0^{occ} all other p_i^{occ}. You can use the notation $\rho := \lambda/\mu$.

Solution

$$p_1 = \rho p_0,$$
$$\text{for } i \geq 2, \quad p_i = p\rho p_{i-1} = (p\rho)^2 p_{i-2} = \ldots = (p\rho)^{i-1} p_1 = p^{i-1}\rho^i p_0.$$

Note that the top formula is compatible with the bottom formula, i.e. we have

$$p_i = p^{i-1}\rho^i p_0 \quad \text{for all } i \geq 1.$$

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(e) Denote by λ_{eff} the effective arrival rate into the system. What is the relation between the average number of customers in the queue and the average waiting time?

Solution $EL^q = \lambda_{\text{eff}} EW$.

(f) Knowing p^{occ}, how would you find the proportion of *lost* customers (that is, of customers who do not join the system because they do not wish to wait)? You can follow the guidelines or provide your own solution.

Guided solution The proportion of arrival attempts that see i customers in the system is p_i^{occ}, due to the PASTA property.

The proportion of arrival attempts that see i customers in the system *and* do not join is $p_i^{occ}(1-p)$. This is relevant for $i \geq 1$.

In total, the proportion of arrival attempts that do not join (that is, the proportion of lost customers) is $\sum_{i=1}^{\infty} p_i^{occ}(1-p) = (1-p)(1-p_o^{occ})$.

My solution

The remaining two questions are OPTIONAL, it is safe to skip them.
Try them if you are done early.

(g) Revisit (d) and calculate p_0^{occ}.

Solution The normalization equation gives

$$1 = \sum_{i=0}^{\infty} p_i = p_0(1 + \sum_{i=1}^{\infty} p^{i-1}\rho^i) = p_0(1 + \rho \sum_{i=1}^{\infty} (p\rho)^{i-1}) = p_0(1 + \rho \sum_{j=0}^{\infty} (p\rho)^j) = p_0(1 + \frac{\rho}{1-p\rho}),$$

hence $p_0^{occ} = \dfrac{1-p\rho}{1-p\rho+\rho}$.

(h) The effective arrival rate is $\lambda_{\text{eff}} = \lambda(p_0^{occ} + (1-p_0^{occ})p)$. Explain why.

Solution Fraction p_0^{occ} of the arrival attempts are to state 0 and all of them join. The remaining fraction $1 - p_0^{occ}$ of the arrival attempts are all to non-zero states, they would have to wait, and hence out of them fraction p join. The total fraction that joins is thus precisely the factor next to λ. Note that it is complementary ("1-") to the fraction of lost customers we found in (f).