Midterm exam Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

October 26, 2022, 12:15–14:30

This exam consists of three exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. The use of books or a graphical calculator is not allowed. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}.$$

(a) [**4pt**] What is the expected number of steps it takes to reach state 5 from state 2?

(b) [**3pt**] What is the probability that it takes *at most* five steps to reach (for the first time) state 5 from state 2?

(c) [**3pt**] What is the probability that it takes *at most* ten steps to reach (for the first time) state 5 from state 2? An analytic-form answer suffices. If you use a matrix power in your answer, fully specify the matrix.

(d) [**5pt**] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether a limit distribution exists and find the limit distribution in case it exists.

(e) [**2pt**] Make the Markov chain irreducible by adding *one* transition arrow and adjusting the transition probabilities if necessary.

Question 2. Every day Bob commutes to work in the morning and then commutes back home in the evening. From time to time he likes to buy a coffee to-go for his commute. As an environmentally conscious person, Bob owns three travel coffee cups and uses them for his to-go coffees when he can.

To be more specific, each commute Bob feels like having a coffee with probability 2/3, independently of his other commutes. If Bob does feel like having a coffee and finds one of the travel cups at his present location, he grabs that cup along (so at the end of that commute the cup ends up at the other location). If Bob does feel like having a coffee and finds no travel cup at his present location, he will buy a coffee in a disposable cup. If Bob does not feel like having a coffee, he does not carry any travel cups along.

(a) [**4pt**] Argue that the sequence

$$X_n = \text{number of travel cups at home at the end of day } n$$

is a discrete-time (time-homogeneous) Markov chain.

In (b) and (c), make sure to specify which one out of π^{occ}, π^{lim} *is relevant.*

(b) [**4pt**] What is the long-run fraction of days on which Bob finds no travel cup at home?

(c) [**5pt**] Each time Bob uses a travel cup, he gets a discount of €0.25 on his coffee. How much does Bob save on average per day over the long time-run? And during a month (a month is 4 weeks, a week is 5 working days)?

Question 3. An online service desk receives two types of tickets. Type-1 (standard) tickets arrive according to a Poisson process $\{N_1(t), t \geq 0\}$ of rate $\lambda_1 = 80$ per hour. Type-2 (special) tickets arrive according to a Poisson process $\{N_2(t), t \geq 0\}$ of rate $\lambda_2 = 20$ per hour. The two arrival processes are independent.

(a) [**4pt**] Which properties of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ ensure that (i) the *total* number of tickets arriving in a time interval $(s, t]$ has a Poisson distribution? (ii) the *total* numbers of tickets arriving in *two* non-overlapping time intervals $(s_1, t_1]$, $(s_2, t_2]$ are independent?

The service desk consists of two teams, A and B. Type-1 tickets are routed, independently of each other, either to team A or to team B with probabilities 3/4 and 1/4, respectively. Type-2 tickets always go to team B.

(b) [3pt] What is the expected time between two consecutive ticket arrivals to team B?

(c) [4pt] What is the joint probability that the following happens during the next 3 minutes: at most 3 tickets arrive to the service desk and no type-1 tickets arrive to team B?

(d) [4pt] (i) What is the probability that the first arrival to the desk is a type-1 ticket for team B? (ii) What is the probability that the first three arrivals to the desk are, in this precise order, a type-1 ticket for team A, a type-1 ticket for team B, and a type-2 ticket (for team B)?