SOLUTIONS
Midterm exam Stochastic Modelling
October 26, 2022

Question 1. Consider a discrete-time Markov chain on the state space $\{1,2,3,4,5,6\}$ with transition matrix

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

(a) [4pt] What is the expected number of steps it takes to reach state 5 from state 2?

Solution Let $T_5 := \min\{n \geq 0\colon X_n = 5\}$ and $m_i := E(T_5 \mid X_0 = i)$. The question is what is m_2. By conditioning on the 1st step we get the system

$$\begin{cases} m_1 = 1 + 2/3 m_2 + 1/3 m_2, & 3)\ m_1 = 3 + m_2 \\ m_2 = 1 + 1/2 m_1 + 1/2 m_3, & 2)\ m_2 = 2 + 1/2 m_1 \quad 4)\ m_2 = 3.5 + 1/2 m_2 \Rightarrow m_2 = 7. \\ m_3 = 1 + m_4, & 1)\ m_3 = 2 \\ m_4 = 1. \end{cases}$$

(b) [3pt] What is the probability that it takes *at most* five steps to reach (for the first time) state 5 from state 2?

Solution

$$P(T_5 \leq 5 | X_0 = 2) = p_{23}p_{34}p_{45} + p_{21}p_{12}p_{23}p_{34}p_{45} = (1 + \tfrac{1}{2} \cdot \tfrac{1}{3}) \cdot \tfrac{1}{2} \cdot 1 \cdot 1 = \tfrac{7}{12}.$$

(c) [3pt] What is the probability that it takes *at most* ten steps to reach (for the first time) state 5 from state 2? An analytic-form answer suffices. If you use a matrix power in your answer, fully specify the matrix.

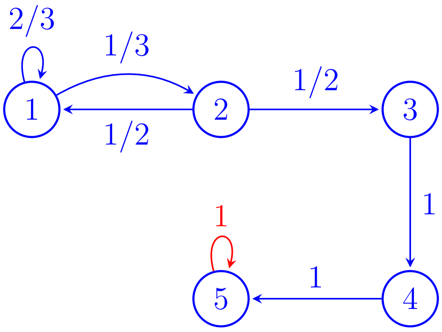
Solution

$$P(T_5 \leq 10 | X_0 = 2) = (\tilde{P}^{10})_{25},$$

where

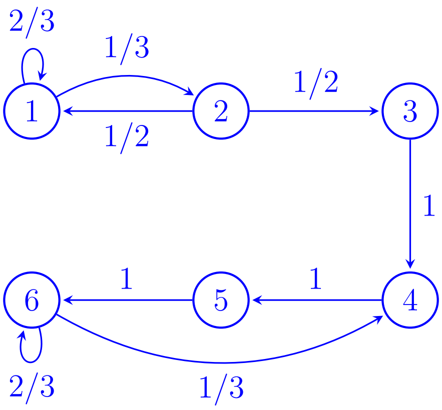
$$\tilde{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

i.e. $\tilde{P}$ is the transition matrix of the adjusted Markov Chain with state 5 made absorbing (compare to the original diagram in (d))



(d) [5pt] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether a limit distribution exists and find the limit distribution in case it exists.

Solution Looking at the transition diagram



there are three communicating classes:

- {1,2} is transient,
- {3} is transient,

- {4,5,6} is absorbing.

Consider the absorbing class {4,5,6} in isolation. It is finite, (irreducible), and aperiodic and hence has the limit distribution (for $X_0 = 4, 5, 6$). To find the limit distribution we solve the system of balance of normalization equations for this class:

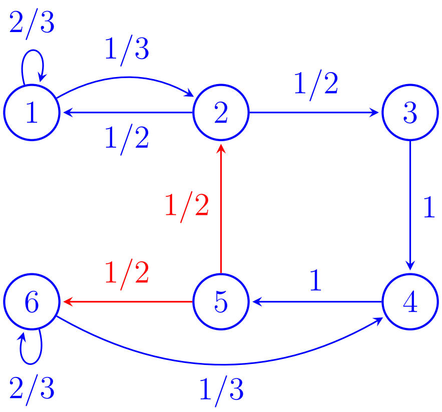
$$\begin{cases} \pi_4 = \pi_6 * 1/3, \\ \pi_5 = \pi_4, \\ \pi_6 * 1/3 = \pi_5, \\ \pi_3 + \pi_4 + \pi_5 = 1. \quad (\frac{1}{3} + \frac{1}{3} + 1)\pi_6 = 1 \Rightarrow \pi_6 = \frac{3}{5} \Rightarrow \pi_4 = \pi_5 = \frac{1}{5} \end{cases}$$

If the MC starts in either of the transient classes {1,2}, {3}, it will end up in the absorbing class eventually. Hence, for any initial state $X_0 = 1, 2, 3, 4, 5, 6$, the limit distribution exists and is the limit distribution on the absorbing class,

$$\pi^{lim} = (0, 0, 0, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}).$$

(e) [2pt] Make the Markov chain irreducible by adding *one* transition arrow and adjusting the transition probabilities if necessary.

Solution For example,



Question 2. Every day Bob commutes to work in the morning and then commutes back home in the evening. From time to time he likes to buy a coffee to-go for his commute. As an environmentally conscious person, Bob owns three travel coffee cups and uses them for his to-go coffees when he can.

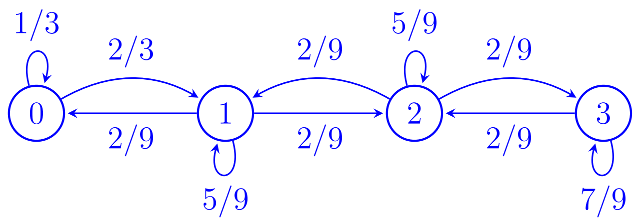
To be more specific, each commute Bob feels like having a coffee with probability 2/3, independently of his other commutes. If Bob does feel like having a coffee and finds one of the travel cups at his present location, he grabs that cup along (so at the end of that commute the cup ends up at the other location). If Bob does feel like having a coffee and finds no travel cup at his present location, he will buy a coffee in a disposable cup. If Bob does not feel like having a coffee, he does not carry any travel cups along.

(a) [**4pt**] Argue that the sequence

$$X_n = \text{number of travel cups at home at the end of day } n$$

is a discrete-time (time-homogeneous) Markov chain.

Solution The transition diagram is



To *explain* how the *transition probabilities* are calculated: there are 4 combinations of coffee-wishes for the commute to work and then the commute back to work, C meaning"Bob wants coffee" and N meaning "Bob does not want coffee",

CC wp $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$, NC wp $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$, CN wp $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$, NN wp $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Eg transitions out of state 1 are the following: $1 \to 2$ in case of NC, $1 \to 0$ in case of CN, and $1 \to 1$ in case of CC or NN.

Note that there is a simpler way to think of transitions out of state 0: they are only determined by what happens on the way back home (as on the way to work in the morning Bob does not carry a travel cup along anyways). If Bob wants coffee on the way back home (wp 2/3) we get a transition $0 \to 1$, and if Bob does not want coffee on the way back home (wp 1/3) we get a transition $0 \to 0$.

The *Markov property* follows from the independence of Bob's coffee wishes for the different commutes. The *time-homogeneity* is in place since the transition probabilities in the diagram do not depend on time n. To summarize,

X_n, $n \geq 0$, is a time-homogeneous DTMC.

In (b) and (c), make sure to specify which one out of π^{occ}, π^{lim} is relevant.

(b) [4pt] What is the long-run fraction of days on which Bob finds no travel cup at home?

Solution The answer to the question is π_0^{occ}. Since the MC is finite and irreducible, the system of balance and normalization equations gives π^{occ}. We have

$$\begin{cases} \pi_0 \frac{2}{3} = \pi_1 \frac{2}{9}, & 1) \pi_1 = 3\pi_0 \\ \pi_1 \frac{4}{9} = \pi_0 \frac{2}{3} + \pi_2 \frac{2}{9}, & 2) \pi_0 \frac{4}{3} = \pi_0 \frac{2}{3} + \pi_2 \frac{2}{9} \Rightarrow \pi_2 = 3\pi_0 \\ \pi_2 \frac{4}{9} = \pi_1 \frac{2}{9} + \pi_3 \frac{2}{9}, & \\ \pi_3 \frac{2}{9} = \pi_2 \frac{2}{9}, & 3) \pi_3 = \pi_2 = 3\pi_0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 & 4) \pi_0 + 3\pi_0 + 3\pi_0 + 3\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{10}, \end{cases}$$

i.e.

$$\pi^{occ} = (\tfrac{1}{10}, \tfrac{3}{10}, \tfrac{3}{10}, \tfrac{3}{10}).$$

The final answer is $\pi_0^{occ} = 1/10$.

(c) [5pt] Each time Bob uses a travel cup, he gets a discount of €0.25 on his coffee. How much does Bob save on average per day over the long time-run? And during a month (a month is 4 weeks, a week is 5 working days)?

Solution The generic discount in each state is:

$$C_0 = \begin{cases} 0.25 & \text{wp } 2/3, \quad \text{if Bob wants coffee on the way back home} \\ 0 & \text{wp } 1/3; \quad \text{if Bob does not want coffee on the way back home} \end{cases}$$

$$C_1 = C_2 = \begin{cases} 0.5 & \text{wp } 4/9, \quad \text{if CC} \\ 0.25 & \text{wp } 4/9, \quad \text{if CN or NC} \\ 0 & \text{wp } 1/9; \quad \text{if NN} \end{cases}$$

$$C_3 = \begin{cases} 0.5 & \text{wp } 4/9, \quad \text{if CC} \\ 0.25 & \text{wp } 2/9, \quad \text{if CN} \\ 0 & \text{wp } 3/9; \quad \text{if NC or NN} \end{cases}$$

Hence the long-run average discount per day is

$$\pi_0^{occ} EC_0 + \pi_1^{occ} EC_1 + \pi_2^{occ} EC_2 + \pi_3^{occ} EC_3 = \frac{1}{10}(\frac{2}{3}\cdot 0.25) + (\frac{3}{10}+\frac{3}{10})(\frac{4}{9}\cdot 0.5 + \frac{4}{9}\cdot 0.25) + \frac{3}{10}(\frac{4}{9}\cdot 0.5 + \frac{2}{9}\cdot 0.25) = €0.3.$$

During a month, Bob saves on average

$$0.3 * 4 * 5 = €6.$$

Question 3. An online service desk receives two types of tickets. Type-1 (standard) tickets arrive according to a Poisson process $\{N_1(t), t \geq 0\}$ of rate $\lambda_1 = 80$ per hour. Type-2 (special) tickets arrive according to a Poisson process $\{N_2(t), t \geq 0\}$ of rate $\lambda_2 = 20$ per hour. The two arrival processes are independent.

(a) [**4pt**] Which properties of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ ensure that (i) the *total* number of tickets arriving in a time interval $(s, t]$ has a Poisson distribution? (ii) the *total* numbers of tickets arriving in *two* non-overlapping time intervals $(s_1, t_1]$, $(s_2, t_2]$ are independent?

Solution (i) We have $N_1(s,t] \sim \text{Poisson}(\lambda_1 t)$, $N_2(s,t] \sim \text{Poisson}(\lambda_2 t)$, $N_1(s,t]$ and $N_2(s,t]$ are independent. By merging of Poisson random variables, the total is $N_1(s,t] + N_2(s,t] \sim \text{Poisson}\big((\lambda_1 + \lambda_2)t\big)$.

(ii) We have $N_1(s_1,t_1]$, $N_1(s_2,t_2]$ independent and $N_2(s_1,t_1]$, $N_2(s_2,t_2]$. Hence th total on one interval $N_1(s_1,t_1] + N_2(s_1,t_1]$ is independent from the total on the other interval $N_1(s_2,t_2] + N_2(s_2,t_2]$.

The service desk consists of two teams, A and B. Type-1 tickets are routed, independently of each other, either to team A or to team B with probabilities 3/4 and 1/4, respectively. Type-2 tickets always go to team B.

(b) [**3pt**] What is the expected time between two consecutive ticket arrivals to team B?

Solution Let
- $N_{1A}(t), t \geq 0$ denote the arrival process of type 1 customers to team A,
- $N_{1B}(t), t \geq 0$ denote the arrival process of type 1 customers to team B.

By *thinning*, the two new processes are both Poisson and their rates are

$$\lambda_{1A} := \lambda_1 * 3/4 = 60 \text{ per hour}, \quad \lambda_{1B} := \lambda_1 * 1/4 = 20 \text{ per hour}.$$

Also the two new processes are independent from each other and from type-2 arrivals $N_2(t), t \geq 0$.

The total arrival process to team B is the *merger process* $N_{1B}(t)+N_2(t), t \geq 0$; it is again Poisson with rate $\lambda_{1B} + \lambda_2 = 20 + 20 = 40$ per hour. Hence, the inter-arrival times to team B are Exponential(40 per hour) with the expectation

$$E\text{Exponential}(40 \text{ per hour}) = \tfrac{1}{40} \text{ hour} = 1.5min.$$

(c) [4pt] What is the joint probability that the following happens during the next 3 minutes: at most 3 tickets arrive to the service desk and no type-1 tickets arrive to team B?

Solution As discussed in (b), the three Poisson processes $N_{1A}(t)$, N_{1B}, $N_2(t)$, $t \geq 0$, are all independent with rates $\lambda_{1A} = 60$, $\lambda_{1B} = 20$, $\lambda_2 = 20$ per hour. We have to find

$$P((N_{1A} + N_{1B} + N_2)(\tfrac{3}{60}) \leq 3, N_{1B}(\tfrac{3}{60}) = 0) = P((N_{1A} + N_2)(\tfrac{3}{60}) \leq 3, N_{1B}(\tfrac{3}{60}) = 0)$$

$$\text{3 processes are independent } = P((N_{1A} + N_2)(\tfrac{3}{60}) \leq 3) * P(N_{1B}(\tfrac{3}{60}) = 0)$$

$$\text{merging in the 1st probability} = P(\text{Poisson}\big((60 + 20)\tfrac{3}{60}\big) \leq 3) * P(\text{Poisson}\big(20\tfrac{3}{60}\big) = 0)$$

$$= e^{-4}(1 + 4 + \frac{4^2}{2} + \frac{4^3}{6}) * e^{-1} = \tfrac{71}{3}e^{-5}.$$

(d) [4pt] What is the probability that the first arrival to the desk is a type-1 ticket for team B? What is the probability that the first three arrivals to the desk are, in this precise order, a type-1 ticket for team A, a type-1 ticket for team B, and a type-2 ticket (for team B)?

Solution Note that the inter-arrival times of type-1 tickets to team A are Exponential($\lambda_{1A} = 60$), the inter-arrival times of type-1 tickets to team B are Exponential($\lambda_{1B} = 20$), the inter-arrival times of type-2 tickets to team B are Exponential($\lambda_2 = 20$). In the 1st question, we want to know

$$P(\text{Exp}(\lambda_{1B}) \text{ wins from } \text{Exp}(\lambda_{1A}), \text{Exp}(\lambda_B)) = \frac{\lambda_{1B}}{\lambda_{1A} + \lambda_{1B} + \lambda_2} = \frac{20}{60 + 20 + 20} = 0.2.$$

In the 2nd question, by the memoryless property, we have three consecutive competitions among three Exponentials (some full, some remaining), and the probability is

$$P(\text{Exp}(\lambda_{1A}) \text{ wins from } \text{Exp}(\lambda_{1B}), \text{Exp}(\lambda_B)) * P(\text{Exp}(\lambda_{1B}) \text{ wins from } \text{Exp}(\lambda_{1A}), \text{Exp}(\lambda_B))*$$

$$*P(\text{Exp}(\lambda_2) \text{ wins from } \text{Exp}(\lambda_{1A}), \text{Exp}(\lambda_{1B})) =$$

$$= \frac{\lambda_{1A}}{\lambda_{1A} + \lambda_{1B} + \lambda_2} \frac{\lambda_{1B}}{\lambda_{1A} + \lambda_{1B} + \lambda_2} \frac{\lambda_2}{\lambda_{1A} + \lambda_{1B} + \lambda_2} = \frac{60}{100} \frac{20}{100} \frac{20}{100} = 0.024.$$