Resit 2 Stochastic Modelling (X_400646)

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Faculty of Science

April 13, 2022, 18:45–21:30

This exam consists of five exercises, for which you can obtain 54 points in total. Your grade will be calculated as (number of points + 6)/6. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

(a) [**5pt**] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether an occupancy and/or limit distribution exists and find the occupancy and/or limit distribution in case it exists.

(b) [**4pt**] Assume the initial state is 1. By conditioning on the first step, find the probability that the Markov chain reaches state 4 for the first time without making a direct transition from 3 to 4.

Question 2. Two teams, A and B, meet each other in a series of games until either of the teams has won three games in a row. Each game results in a draw with probability 0.1, team A winning with probability 0.6, or team B winning with probability 0.3. The outcomes of the games are independent.

(a) [5pt] Formulate a discrete-time Markov chain that is suitable to analyse the duration of the game series. Provide the transition diagram rather than the transition matrix.

(b) [3pt] What is the probability that the series takes *at most* 5 games? An analytic-form answer in terms of the transition matrix P suffices. You do not have to provide P itself.

(c) [3pt] Give a system of equations that determines the expected duration of the game series. You do not have to solve this system.

Question 3. Alarms arrive at an emergency desk according to a Poisson process at rate 10 per 24-hour day. Each alarm turns out to be a *false* one with probability 0.1, independently of the other alarms.

(a) [3pt] What is the probability that it takes more than 8 hours till the next *true* alarm?

(b) [5pt] A 24-hour day consists of three 8-hour shifts. What is the probability that exactly 3 *true* alarms are received on a given day but none of them is received in the middle shift of the day?

Question 4. Consider a single-server system where *potential* customers arrive according to a Poisson process with rate λ. If, upon arrival, a customer finds $i = 1, 2, \ldots$ other customers in the system, then this customer joins the queue with probability $1/(i + 1)$ or leaves immediately with probability $i/(i+1)$. A customer that arrives into an empty system immediately proceeds to the server. The service times are distributed exponentially with rate μ.

(a) [6pt] Argue that the number of customers in the system is a continuous-time Markov chain. For which λ and μ is this system stable? Determine the occupancy distribution p^{occ} and limit distribution p^{lim} in the stable scenario.

(b) [4pt] (i) What is the fraction of time the server is idling based on your answer in (a)? (ii) What is the fraction of time the server is idling in terms of the busy period of the server? (iii) Find the average busy period of the server using (i) and (ii).

(c) [3pt] Express the fraction of lost customers in terms of the probabilities p_i^{occ}. You do not have to plug in the solution from (a) and further work out the formula.

Question 5. A service desk handles high- and low-priority customers which arrive according to two independent Poisson processes; the rates are $\lambda_H = 1$ and $\lambda_L = 3$, respectively. There is a single server that serves customers one at a time and the order is as follows: at the end of a service, high-priority customers have priority over low-priority customers, but an ongoing service is never interrupted. Within the high-priority class, the order of service is FIFO; and within the low-priority class the order of service is FIFO. All service times are distributed exponentially with rate $\mu = 8$, regardless of the customer priority.

(a) [**4pt**] What is the average waiting time EW across all of the customers, high- and low- priority together? In particular, is the Pollaczek-Khinchine formula applicable?

(b) [**5pt**] Determine the average waiting time EW_H of high-priority customers and the average number EL_H^q of high-priority customers in the queue by doing Mean Value Analysis for this type of customers.

(c) [**4pt**] Now consider the total number of customers in the system, i.e., neglect the priorities. Also assume that now an idle server requires some *start-up time*. More specifically, when a customer arrives into an empty system, the server does not start service immediately but remains idle for an extra period of time, which is referred to as a start-up time. Assume that the start-up times are distributed exponentially with rate θ. Formulate a continuous-time Markov chain where the state has two components, one of which is the total number of customers in the system.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu \, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u) \, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho) \sum_{i=0}^{c-1} (c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W \, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda \, E(B) = c\rho.$$