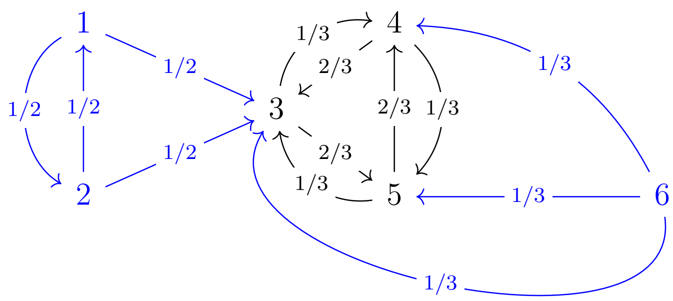
SOLUTIONS
2nd resit Stochastic Modelling
April 13, 2022

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

(a) [**5pt**] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether an occupancy and/or limit distribution exists and find the occupancy and/or limit distribution in case it exists.

Solution Looking at the transition diagram,



there are three communicating classes: transient {1,2}, absorbing {3,4,5}, and transient {6}.

The absorbing class {3,4,5} is finite and aperiodic, and hence the occupancy and limit distribution *within this class* exist for any initial state. They are determined by the balance and normalization equations

$$\begin{cases} (3) \quad \pi_3 = \pi_4 * 2/3 + \pi_5 * 1/3, \\ (4) \quad \pi_4 = \pi_3 * 1/3 + \pi_5 * 2/3, \\ \qquad \pi_5 = \pi_3 * 2/3 + \pi_4 * 1/3, \\ \qquad \pi_3 + \pi_4 + \pi_5 = 1. \end{cases}$$

(4) plugged into (3) gives

$$\pi_3 = \pi_3 * 2/9 + \pi_5 * 4/9 + \pi_5 * 1/3 \quad \Leftrightarrow \quad 7/9\pi_3 = 7/9\pi_5 \quad \Leftrightarrow \quad (5)\ \pi_3 = \pi_5,$$

and (5) plugged into (3) gives

$$\pi_3 = \pi_4 * 2/3 + \pi_3 * 1/3 \quad \Leftrightarrow \quad \pi_3 = \pi_4.$$

Now normalization implies $\pi_3 = \pi_4 = \pi_5 = 1/3$.

Finally, for any initial state $X_0 = 1, 2, \dots, 6$, the Markov chain ends up in the absorbing class $\{3, 4, 5\}$, and hence the limit and occupancy distributions exist and are given by

$$\pi^{lim} = \pi^{occ} = (0, 0, 1/3, 1/3, 1/3, 0, 0).$$

(b) [**4pt**] Assume the initial state is 1. By conditioning on the first step, find the probability that the Markov chain reaches state 4 for the first time without making a direct transition from 3 to 4.

Solution The question is what is q_1, where

$$q_i := P(\text{reach 4 for the 1st time without a direct transition } 3 \to 4 \mid X_0 = i).$$

By conditioning on the 1st step we get the system

$$\begin{cases} (1) \quad q_1 = 1/2q_2 + 1/2q_3, \\ (2) \quad q_2 = 1/2q_1 + 1/2q_3, \\ (3) \quad q_3 = 1/3 * 0 + 2/3q_5, \\ (5) \quad q_5 = 1/3q_3 + 2/3 * 1. \end{cases}$$

Since (1) and (2) have the same RHS, it follows that $q_1 = q_2$ and then from (1) it also follows that $q_1 = q_3$. (5) plugged into (3) gives

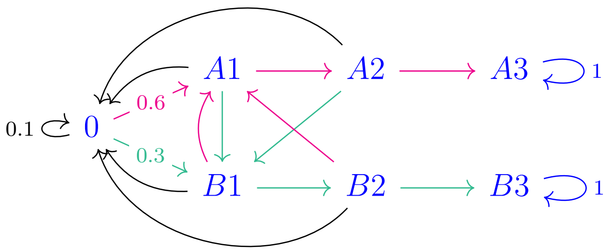
$$q_3 = 2/9q_3 + 4/9 \quad \Leftrightarrow \quad q_3 = 4/7.$$

To conclude, the answer is $q_1 = 4/7$.

Question 2. Two teams, A and B, meet each other in a series of games until either of the teams has won three games in a row. Each game results in a draw with probability 0.1, team A winning with probability 0.6, or team B winning with probability 0.3. The outcomes of the games are independent.

(a) **[5pt]** Formulate a discrete-time Markov chain that is suitable to analyse the duration of the game series. Provide the transition diagram rather than the transition matrix.

Solution Let $X_n = 0$ if game n results in a draw and otherwise $X_n =$ (winning team in game n, current # consecutive wins by this team). This is a DTMC with transition diagram



(with all ).

The *Markov property* follows from the independence of the games. We also have the *time-homogeneity* as the probabilities for A to win, B to win, or a draw are the same for all games n.

(b) **[3pt]** What is the probability that the series takes *at most* 5 games? An analytic-form answer in terms of the transition matrix P suffices. You do not have to provide P itself.

Solution The series is over as soon as state $3A$ or state $3B$ is reached for the 1st time, i.e. the series duration is $T = \min\{n \geq 1 \colon X_n = A3 \text{ or } B3\}$. The probability in question is

$$P(T \leq 5 \mid X_0 = 0) \overset{A3,\,B3 \text{ absorbing states}}{=} P(X_5 = A3 \text{ or } X_5 = B3 \mid X_0 = 0)$$
$$= (P^5)_{0,A3} + (P^5)_{0,B3}.$$

(c) [3pt] Give a system of equations that determines the expected duration of the game series. You do not have to solve this system.

Solution We want to know m_0, where $m_i = E(T \mid X_0 = i)$. The m_i's are determined by the system

$$\begin{cases} m_0 &= 1 + 0.1m_0 + 0.6m_{A1} + 0.3m_{B1}, \\ m_{A1} &= 1 + 0.1m_0 + 0.6m_{A2} + 0.3m_{B1}, \\ m_{B1} &= 1 + 0.1m_0 + 0.6m_{A1} + 0.3m_{B2}, \\ m_{A2} &= 1 + 0.1m_0 + 0.6 * 0 + 0.3m_{B1}, \\ m_{B2} &= 1 + 0.1m_0 + 0.6m_{A1} + 0.3 * 0. \end{cases}$$

Question 3. Alarms arrive at an emergency desk according to a Poisson process at rate 10 per 24-hour day. Each alarm turns out to be a *false* one with probability 0.1, independently of the other alarms.

(a) [3pt] What is the probability that it takes more than 8 hours till the next *true* alarm?

Solution By thinning, the true alarms arrive according to a Poisson process at rate $\lambda = 0.9 * 10 = 9$ per 24-hour day. Hence, the time till the next true alarm has an Exponential distribution with this rate and the answer is

$$P(\text{Exp}(\lambda) > 8/24) = e^{-9*8/24} = e^{-3}.$$

(b) [5pt] A 24-hour day consists of three 8-hour shifts. What is the probability that exactly 3 *true* alarms are received on a given day but none of them is received in the middle shift of the day?

Solution Denote by $N(\cdot)$ the arrival process of the true alarms (it is a Poisson process with rate $\lambda = 9$). The question is what is

$$P\{\overbrace{N(2/3) - N(1/3)}^{\#\text{ true alarms in the middle shift}} = 0, \overbrace{\Big(N(1/3) - N(0)\Big) + \Big(N(1) - N(2/3)\Big)}^{\#\text{ true alarms in the 1st and 3rd shifts together}} = 3\}?$$

We note that
- the numbers of true alarms in the three shifts are independent because the shifts do not overlap,
- the number of true alarms in each of the three shifts $\sim \text{Poi}(\lambda*1/3) = \text{Poi}(3)$;

- due to independence and merging of Poisson random variables, the number of true alarms in the 1st and 3rd shift together $\sim \text{Poi}(3+3) = \text{Poi}(6)$.

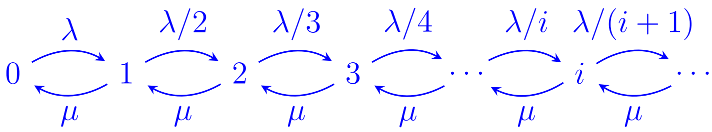
Hence,the answer is

$$e^{-3} * e^{-6} 6^3/3! = 36e^{-9}.$$

Question 4. Consider a single-server system where *potential* customers arrive according to a Poisson process with rate λ. If, upon arrival, a customer finds $i = 1, 2, \ldots$ other customers in the system, then this customer joins the queue with probability $1/(i+1)$ or leaves immediately with probability $i/(i+1)$. A customer that arrives into an empty system immediately proceeds to the server. The service times are distributed exponentially with rate μ.

(a) [6pt] Argue that the number of customers in the system is a continuous-time Markov chain. For which λ and μ is this system stable? Determine the occupancy distribution p^{occ} and limit distribution p^{lim} in the stable scenario.

Solution Let $L(t)$ be the number of customers in the system at time t, $t \geq 0$. This is a CTMC since all transitions take Exponential times according to the diagram



Stability condition: Intuitively, this CTMC is stable for all λ and μ because, in large states, the growth rate λ/i is below the decay rate μ. Formally, this (irrducible) CTMC is stable because there exists a solution to balance and normalization equations for any λ and μ, as we show below. This solution is both p^{lim} and p^{occ}.

The system for p^{lim} and p^{occ} is,

$$\begin{cases} \text{global balance for sets } \{0, \ldots, i-1\}: \\ p_{i-1} * \lambda/i = p_i * \mu, \quad i = 1, 2, \ldots \\ \text{normalization: } \sum_{i=0}^{\infty} p_i = 1. \end{cases}$$

It follows that,

$$p_i = \frac{\lambda/\mu}{i} p_{i-1} = \frac{\lambda/\mu}{i} \frac{\lambda/\mu}{i-1} p_{i-2} = \frac{\lambda/\mu}{i} \frac{\lambda/\mu}{i-1} \frac{\lambda/\mu}{i-2} p_{i-3}$$
$$= \ldots = \frac{\lambda/\mu}{i} \frac{\lambda/\mu}{i-1} \frac{\lambda/\mu}{i-2} \cdots\cdots \frac{\lambda/\mu}{1} p_0 = \frac{(\lambda/\mu)^i}{i!} p_0, \quad i \geq 0,$$

which we plug into the normalization equation and get

$$1 = p_0 \sum_{i=0}^{\infty} \frac{(\lambda/\mu)^i}{i!} = p_0 e^{\lambda/\mu} \quad \Leftrightarrow \quad p_0 = e^{-\lambda/\mu}.$$

Hence,

$$p_i^{occ} = p_i^{lim} = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0.$$

(b) [**4pt**] (i) What is the fraction of time the server is idling based on your answer in (a)? (ii) What is the fraction of time the server is idling in terms of the busy period of the server? (iii) Find the average busy period of the server using (i) and (ii).

Solution (i) % time server idling $= p_0^{occ} = e^{-\lambda/\mu}$.

(ii) The server alternates between periods of idling and busy periods. Idling is waiting for a customer to arrive into an empty system and takes an Exponential(λ) amount of time. Hence

$$\%\text{ time server idling} = \frac{E(\text{idling})}{\underbrace{E(\text{idling})}_{1/\lambda} + E(BP)}.$$

(iii) We have

$$e^{-\lambda/\mu} = \%\text{ time server idling} = \frac{1/\lambda}{1/\lambda + E(BP)},$$

which gives

$$e^{-\lambda/\mu}\frac{1}{\lambda} + e^{-\lambda/\mu} E(BP) = \frac{1}{\lambda} \quad \Leftrightarrow \quad E(BP) = \frac{1}{\lambda} * \frac{1 - e^{-\lambda/\mu}}{e^{-\lambda/\mu}} = \frac{e^{\lambda/\mu} - 1}{\lambda}.$$

(c) [**3pt**] Express the fraction of lost customers in terms of the probabilities p_i^{occ}. You do not have to plug in the solution from (a) and further work out the formula.

Solution By PASTA, fraction p_i^{occ} of potential customers upon their arrival find i other customers already in the system. Out of customers who, upon arrival, find i other customers already in the system, fraction $i/(i+1)$ leaves immediately (i.e. they are lost). Hence, in total the fraction of lost customers is

$$\sum_{i=1}^{\infty} p_i^{occ} \frac{i}{i+1}.$$

Question 5. A service desk handles high- and low-priority customers which arrive according to two independent Poisson processes; the rates are $\lambda_H = 1$ and $\lambda_L = 3$, respectively. There is a single server that serves customers one at a time and the order is as follows: at the end of a service, high-priority customers have priority over low-priority customers, but an ongoing service is never interrupted. Within the high-priority class, the order of service is FIFO; and within the low-priority class the order of service is FIFO. All service times are distributed exponentially with rate $\mu = 8$, regardless of the customer priority.

(a) [**4pt**] What is the average waiting time EW across all of the customers, high- and low- priority together? In particular, is the Pollaczek-Khinchine formula applicable?

Solution The total number of customers is an $M/M/1$ queue that is stable (the arrival rate $\lambda_H + \Lambda_L <$ service rateμ) and has a work-conserving non-preemptive service discipline. Hence the PK formula is indeed applicable and we have

$$EW = \frac{\rho}{1-\rho} ER,$$

where

$$\rho = (\lambda_H + \lambda_L) * \frac{1}{\mu} = 4/8 = 1/2,$$
$$\text{residual service time } ER = E(\text{Exp}(\mu)) = 1/8,$$

and that gives $EW = 1/8$.

(b) [**5pt**] Determine the average waiting time EW_H of high-priority customers and the average number EL_H^q of high-priority customers in the queue by doing Mean Value Analysis for this type of customers.

Solution The MVA equations are:

$$\begin{cases} \text{Little's law} & EL_H^q \;\; = \lambda_H EW_H, \\ \text{arrival relation} & EW_H \;\; = \rho * E\text{Exp}(\mu) + EL_H^q * E\text{Exp}(\mu). \end{cases}$$

The arrival relation above comes up as follows:
- By PASTA, it is with probability ρ that a high-priority customer arrives while the server is busy and has to wait for the residual service time at the server, which is Exp(μ) regardless of whether it is a low- or high- priority customer at the server.
- Each high-priority customer has to wait for the full service times of the high-priority customers in front of him in the queue.

We plug in the rates and ρ and get

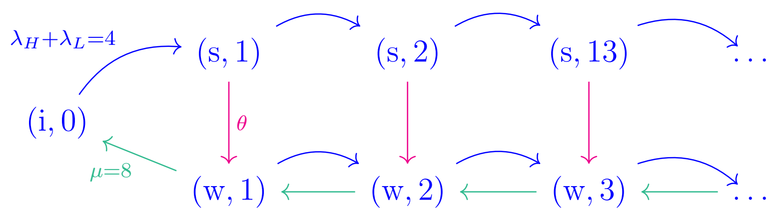
$$\begin{cases} \text{Little's law} & EL_H^q \;\; = EW_H, \\ \text{arrival relation} & EW_H \;\; = \frac{1}{2} * \frac{1}{8} + EW_H * \frac{1}{8} \end{cases} \quad \Rightarrow \quad EW_H = \frac{1}{14} = EL_H^q.$$

(c) [4pt] Now consider the total number of customers in the system, i.e., neglect the priorities. Also assume that now an idle server requires some *start-up time*. More specifically, when a customer arrives into an empty system, the server does not start service immediately but remains idle for an extra period of time, which is referred to as a start-up time. Assume that the start-up times are distributed exponentially with rate θ. Formulate a continuous-time Markov chain where the state has two components, one of which is the total number of customers in the system.

Solution Let

$$X(t) = (\text{'i'dling/'s'tarting-up/'w'orking state of the server}, \\ \#\text{ customers in the system}) \quad \text{at time } t.$$

This is a CTMC with transition diagram



(with all $\xrightarrow{4}$, $\xrightarrow{8}$, $\xrightarrow{\theta}$).

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$