Resit Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

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This exam consists of five exercises, for which you can obtain 54 points in total. Your grade will be calculated as (number of points + 6)/6. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5\}$.

(a) [5pt] Assume the transition matrix is

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

For initial state (i) $X_0 = 1$, determine with which probability the Markov chain ends up in each of the absorbing classes. For initial states (ii) $X_0 = 2$, (iii) $X_0 = 5$, determine whether a limit distribution exists and find the limit distribution in case it exists.

(b) [3pt] Assume the transition matrix is

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Note that the only difference with part (a) is in the transitions out of state 3. What is the expected number of steps it takes to reach state 4 from state 1?

Question 2. A furniture store carries in its range a certain type of bed for which the monthly demands are independent and distributed uniformly between 0 and 4 (i.e., the demand during each month is i with probability $1/5$ for $i = 0, 1, \ldots, 4$). Due to limited storage, the store can have up to 4 beds in stock at a time. The demand that arrives while beds are out of stock is lost. The inventory control is as follows. If there are less than 2 beds in stock at the end of a month, then the shop restocks up to full capacity and starts the next month with 4 beds in stock (the delivery time from the supplier to the shop is negligible). Otherwise the shop does not restock and starts the next month with what is left from the previous month.

(a) [5pt] Argue that the inventory levels at the end of each month form a discrete-time Markov chain. Provide the transition matrix and a system of equations that determines the occupancy and limit distribution. Without solving this system, argue why the occupancy and limit distributions exist.

(b) [3pt] If the distribution from part (b) were known, how would you calculate the long-run average number of lost sales per month?

(c) [5pt] Consider a new situation where customers that arrive while beds are out of stock are not lost. Instead backorders are placed for these customers. *At the end of a month with backorders*, the store orders (and immediately receives) 4 beds from the supplier. Out of these 4 beds, the backorders are immediately covered. The remaining beds are the stock that the store starts the next month with. *For months without backorders*, the replenishment rule is the same as before. Model this new situation as a discrete-time Markov chain assuming additionally that, when the shop opened for the very first time, there was a full stock of 4 beds.
Hint: the final assumption guarantees that the number of backorders never exceeds a certain threshold.

Question 3. In a three-component computer system, the components A, B, and C work in parallel and experience failures. A failed component is replaced immediately. All lifetimes are independent and distributed exponentially. The rates are $\lambda_A = 2$, $\lambda_B = 1$, $\lambda_C = 1$ per day for components of type A, B, C, respectively.

(a) [5pt] What is the joint probability that the following happens on a given day: there are two failures, both of them happen in the second half of the day, and none of them is of component A?

(b) [3pt] What is the probability that the first failure in this system is of component A?

(c) [3pt] What is the joint probability that the first, second and third failure in this system are of component A, B and C, respectively?

Question 4. Consider a stable $M/M/2$ system with arrival rate λ, service rate μ, and the load per server $\rho := \lambda/(2\mu) < 1$.

(a) [6pt] Argue that the number of customers in the system is a continuous-time Markov chain. Determine the occupancy and limit distribution. In particular, derive that

$$p_0^{occ} = p_0^{lim} = \frac{1-\rho}{1+\rho}.$$

(b) [2pt] Express the fraction Π_W of customers that experience waiting in terms of the probabilities p_i^{occ}. You do not have to plug in the solution from (a) and further work out the formula.

(c) [5pt] Use Mean Value Analysis to find the customer-average waiting time EW and the time-average number of customers in the queue EL^q.
Remark: a reference to the formula sheet for EW will not suffice. The waiting probability Π_W can be left in the answer as is.

Question 5. A service desk handles two types of customers that arrive according to two independent Poisson processes, both of unit rate, $\lambda_1 = \lambda_2 = 1$. The service times are distributed exponentially, with rates $\mu_1 = 3$ and $\mu_2 = 6$ for the two customer types, respectively. The customers are helped one at a time in the order of arrival. The waiting room is unlimited.

(a) [4pt] The number of customers present can be viewed as an $M/G/1$ model. Specify the arrival rate and the service time distribution in this $M/G/1$ model, and find the customer-average waiting time.
Reminder: for a random variable $X \sim$ Exponential(α), the variance is $1/\alpha^2$.

(b) [**5pt**] Model the situation at the service desk as a continuous-time Markov chain where the state has two components one of which is the number of customers present.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)}\, e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$