Final exam Stochastic Modelling (X_400646)

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This exam consists of three exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points $+ 5)/5$. The use of books or a graphical calculator is not allowed. At the end of the exam, a formula sheet is included; you can use these formulas without proof if relevant. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. In an auto repair shop, there is a single repairman and room for (at most) two cars at a time (one undergoing repair and one waiting for repair). Cars drop by this repair shop according to a Poisson process of rate 2. A car that drops by and finds the shop already occupied by two other cars can not be taken in and leaves immediately. The repair times of the different cars are independent and distributed exponentially with rate 2. The order of repair is the order of arrival.

(a) [**2pt**] Formulate a CTMC based on which you can answer the subsequent parts of the question.

(b) [**2pt**] At a certain moment the repair shop is empty. What is the expected time until it is full?

(c) [**3pt**] What is the fraction of time that each of the following three situations occur: (i) the shop is empty, (ii) there is exactly one car in the shop, (iii) the shop is full?

(d) [**2pt**] What fraction of cars drop by a full shop and have to leave without repair?

(e) [**2pt**] What is the time-average number of cars in the shop?

Question 2. Customers arrive at a service facility according to a Poisson process of rate λ. Their service times are independent and distributed exponentially with mean $1/\mu$. The number of servers depends on how crowded it is in the system relative to a certain threshold K. There is one *permanent* server and a virtually unlimited amount of *support servers.* At all times when there are K or less customers in the system, the permanent server is handling them on its own, one at a time in the order of arrival. At all times when there are more than K customers in the system, the support servers are involved so that each customer is served at its dedicated server. (Mind that as soon as the number of customers drops down to K, the support is discontinued and it is the permanent server on its own again.)

(a) [4pt] Argue that the number of customers in the system is a CTMC. Argue *intuitively* whether it is a stable CTMC.

(b) [5pt] Find the limit and occupancy distribution. In particular, derive that

$$p_0^{lim} = p_0^{occ} = \left[\frac{1-\rho^{K+1}}{1-\rho} + K! \left(e^{\rho} - \sum_{i=0}^{K} \frac{\rho^i}{i!} \right) \right]^{-1},$$

where $\rho := \lambda/\mu$.

(c) [4pt] The higher the threshold K, the higher the fraction of customers that experience waiting times; *you can use this monotonicity fact without proof.* The table below provides the occupancy and limit probabilities for $\lambda = 5.5$, $\mu = 1$ and a few different values of K. These probabilities are rounded. Based on this table, if $\lambda = 5.5$ and $\mu = 1$, what is the biggest K under which the fraction of customers that experience waiting does not exceed 5%?

	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7
$K=2$	0.002	0.011	0.063	0.115	0.158	0.174	0.159	0.125
$K=3$	0.001	0.004	0.022	0.121	0.166	0.183	0.168	0.132
$K=4$	0.0002	0.001	0.006	0.034	0.187	0.206	0.189	0.148
$K=5$	0.00005	0.0002	0.002	0.008	0.046	0.252	0.231	0.181
$K=6$	0.00001	0.00006	0.0003	0.002	0.01	0.056	0.311	0.244

(d) [5pt] Consider a new situation where there is a different procedure to involve the support servers. It is not the case anymore that the support servers necessarily get involved as soon as the number of customers exceeds K. With each new arrival that leads to more than K customers in the system, the permanent server makes a request for support. The request is satisfied with probability α independently of all previous such requests. In case the request is satisfied, the support servers get involved immediately (so that each customer is served at its dedicated server) and remain involved until the number of customers drops down to K (then it is the permanent server on its own again). In case the request is not satisfied, the permanent server remains on its own until the next opportunity to request support (i.e. until the next arrival that leads to more than K customers in the system).

Formulate a CTMC that models this new situation and, in particular, keeps track of the number of customers in the system. Do you have to include any additional information in the state?

Question 3. There are two communication channels. Channel A handles type A messages, which are generated at rate $1/4$ and are all of fixed size 3. Channel B handles type B messages, which are generated at rate $1/30$ and are varied in size, the size distribution is (approximately) Normal with mean 18 and variance 9. The rates are per second and the sizes are transmission times in seconds at a unit transmission speed. Both channels transmit messages at a unit speed. There is an option of merging the two channels with the benefit of doubling the transmission speed and cutting transmission times by half.

(a) [5pt] Both channels transmit messages in the order in which the messages are generated. In particular, transmissions are delayed on average by 4.5 seconds at channel A and by 13.875 seconds at channel B *(these two values you can use without derivation). Show* that, after merging, the average transmission delay would be approximately 5.13 seconds. Is there an improvement to the average transmission delay of type A messages, the average transmission delay of type B messages, or the weighted average delay?

(b) [3pt] For messages of type A, would their average sojourn time (from the moment a message is generated till its transmission is finished) improve after merging? And for messages of type B?

(c) [3pt] Will your answers to (a) and (b) change in case the order of transmission is non-preemptive LIFO presently and will remain such after merging?
Hint: no new calculation is required.

(d) [5pt] Will your answers to (a) and (b) change in case (the order of transmission is FIFO presently and will remain such after merging but) after merging there are *start-up delays* that are distributed exponentially with mean $1/\theta = 8$ seconds? In order to answer this question, do Mean Value Analysis for the average transmission delay and average number of delayed transmissions. *You can use without proof the fact that the system is stable and the fraction of time the channel is transmitting equals the load.*

To clarify: when, after a period of time with no messages to transmit, a new message is generated, the channel does not start transmitting immediately but remains idle for an additional period of time. This additional idling time is a start-up delay.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual time till next event. Let X be a generic inter-event time and R the residual time till next event. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W under FIFO and the busy period BP under work-conserving disciplines satisfy

$$E(W) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B), \quad \text{where } c_B^2 = \frac{V(B)}{(E(B))^2}$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$