Midterm exam Stochastic Modelling (X_400646)

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This exam consists of four exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points $+ 5)/5$. The use of books or a graphical calculator is not allowed. Please write your name and student number on every paper you hand in and **provide a clear motivation** of all your answers. Good luck!

Question 1. **[4pt]** Let X_n, $n = 0, 1, 2, \ldots$, be a discrete-time Markov chain on a state space S with transition probabilities p_{ij}, $i, j \in S$. Prove that, for all states $i_0, i_1, i_2, i_3 \in S$,

$$P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} p_{i_2 i_3}.$$

Hint: Why is it the case that

$$\begin{aligned} &P(X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &= P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0)? \end{aligned}$$

Use this fact as an inspiration for your proof.

Question 2. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

(a) **[8pt]** For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether a limit distribution exists and find the limit distribution in case it exists.

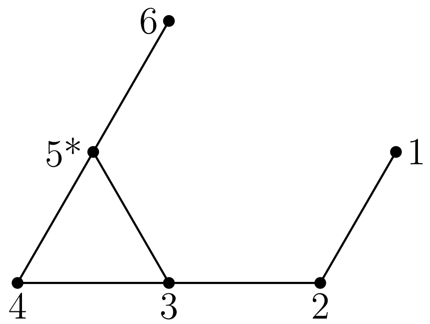
(b) [3pt] What is the expected number of steps it takes to reach state 5 from state 4?

(c) [3pt] What is the probability that it takes *no more than* two steps to reach state 5 from state 4?

(d) [4pt] What is the probability that it takes *more than* ten steps to reach state 5 from state 4? An analytic-form answer suffices.

Question 3. You navigate the city by an electric rental car following the map below. The nodes 1, ..., 6 are the parking lots where you can pick the car at the start of a rental or leave the car at the end of a rental. Having started your rental at a specific location, you end the rental and park the car at one of the *neighbouring* locations, either of them equally likely. For the next rental you pick the car up from the location where you left it last time.

To clarify: neighbouring locations are those directly connected by an edge, e.g. location 4 has two neighbours: 3 and 5*; location 3 has three neighbours: 2, 4, and 5*.



For each rental you are charged a fixed amount €c. In addition, you are charged extra or get a bonus depending on where you park the car at the end of the rental. Location 5 is also a charging station. When you leave the car at location 5, you get a bonus of €d. When you leave the car at any other location, you are charged proportionally to the *distance* to location 5, €k per unit of distance.

To clarify: the *distance* between two locations, in units, is the length of the shortest path, in edges. E.g. the distance between locations 2 and 5* is 2 units, corresponding to the path 2-3-5* (not 3 units corresponding to the path 2-3-4-5*).

(a) [**4pt**] Formulate a discrete-time Markov chain that is suitable to analyse the long-run average costs per rental.

(b) [**7pt**] Calculate the average costs per rental in the long-run.
Hint: from the balance equations for π_i, $i = 1, \ldots, 4$, it follows that

$$\pi_2 = 2\pi_1, \quad \pi_3 = 3\pi_1, \quad \pi_4 = 2\pi_1;$$

you can use these relations without derivation.

Question 4. A service desk serves two types of customers, A and B, which arrive according to independent Poisson processes with rates $\lambda_A = 1$ per hour and $\lambda_B = 3$ per hour. Each customer, of type A or B, independently of the others, will require a follow-up service with probability p.

(a) [**2pt**] What is the probability that it takes longer than 20 minutes between two consecutive arrivals of type B?

(b) [**5pt**] What is the joint probability that the following happens during the first working hour: in the first half-hour at most the expected (total) number of customers arrive and in the second half-hour no customers arrive that will need a follow-up service?
Reminder: For a random variable $X \sim \text{Poisson}(\lambda)$, $EX = \lambda$.

(c) [**5pt**] What is the probability that the second customer of type A arrives before the second customer of type B?