SOLUTIONS
Midterm exam Stochastic Modelling
October 27, 2021

Question 1. [**4pt**] Let X_n, $n = 0, 1, 2, \ldots$, be a discrete-time Markov chain on a state space S with transition probabilities p_{ij}, $i, j \in S$. Prove that, for all states $i_0, i_1, i_2, i_3 \in S$,

$$P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} p_{i_2 i_3}.$$

Hint: Why is it the case that

$$\begin{aligned} &P(X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &= P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0)? \end{aligned}$$

Use this fact as an inspiration for your proof.

Solution We have

$$\begin{aligned} &P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \\ &\overset{(1)}{=} P(X_3 = i_3 \mid X_2 = i_2, X_1 = i_1, X_0 = i_0) P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0) \\ &\overset{(2)}{=} P(X_3 = i_1 \mid X_2 = i_2) P(X_2 = i_1 \mid X_1 = i_1) P(X_1 = i_1 \mid X_0 = i_0) \\ &= p_{i_2 i_3} p_{i_1 i_2} p_{i_0 i_1}, \end{aligned}$$

where (2) is by the Markov property and (1) follows from the definition of conditional probability.

In more detail, (1) is true because

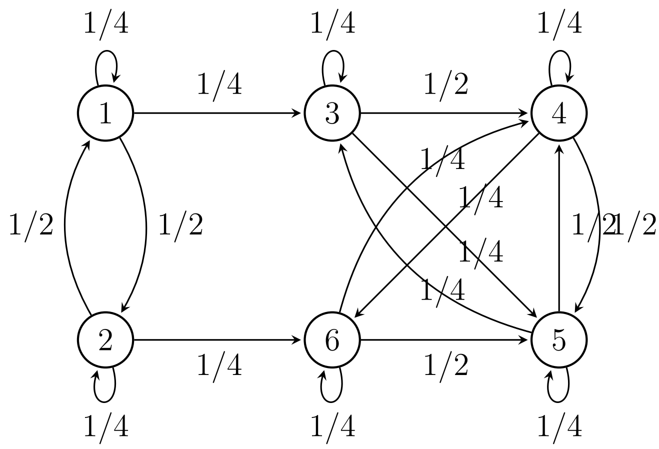
$$\begin{aligned} &P(X_3 = i_3 \mid X_2 = i_2, X_1 = i_1, X_0 = i_0) P(X_2 = i_2 \mid X_1 = i_1, X_0 = i_0) P(X_1 = i_1 \mid X_0 = i_0) \\ &= \frac{P(X_3 = i_3, X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_2 = i_2, X_1 = i_1, X_0 = i_0)} \times \frac{P(X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_1 = i_1, X_0 = i_0)} \times \frac{P(X_1 = i_1, X_0 = i_0)}{P(X_0 = i_0)} \\ &= \frac{P(X_3 = i_3, X_2 = i_2, X_1 = i_1, X_0 = i_0)}{P(X_0 = i_0)} = P(X_3 = i_3, X_2 = i_2, X_1 = i_1 \mid X_0 = i_0) \end{aligned}$$

Question 2. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

(a) [8pt] For each initial state $X_0 = i$, $i = 1, 2, \ldots, 6$, determine whether a limit distribution exists and find the limit distribution in case it exists.

Solution Looking at the transition diagram,



there are two communicating classes:

- {1,2} is transient,
- {3,4,5,6} is absorbing and aperiodic.

Let's consider the absorbing class {3,4,5,6} in isolation. The fact that this class is finite ensures that we can solve the balance and normalization equations; and the aperiodicity ensures that this solution will be the limit distribution for any initial state (from {3,4,5,6}). The balance and normalization equations for this class are:

$$\begin{cases} (1) & \pi_3 * 3/4 = \pi_5 * 1/4, \\ (2) & \pi_4 * 3/4 = \pi_3 * 1/2 + \pi_5 * 1/2 + \pi_6 * 1/4, \\ (3) & \pi_5 * 3/4 = \pi_3 * 1/4 + \pi_4 * 1/2 + \pi_6 * 1/2, \\ (4) & \pi_6 * 3/4 = \pi_4 * 1/4, \\ (5) & \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1. \end{cases}$$

From (1) and (4) it follows that

$$\begin{aligned} & \pi_5 = 3\pi_3 \\ (*) \quad & \pi_4 = 3\pi_6, \end{aligned}$$

we plug these into (2) and get

$$\begin{aligned} \pi_4 * 3/4 &= \pi_3 * 1/2 + \pi_3 * 3/2 + \pi_4 * 1/12, \\ \pi_4 * 8/12 &= \pi_3 * 2, \\ \pi_4 &= 3\pi_3, \\ \pi_6 &\overset{(*)}{=} \pi_4/3 = \pi_3. \end{aligned}$$

The blue equations give everything in terms of π_3. As we plug them into the normalization equation (5), we get the following solution to the system (1)-(5):

$$(\pi_3, \pi_4, \pi_5, \pi_6) = (\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}).$$

Now, if the Markov chain starts in the absorbing class, ie if $X_0 = 3, 4, 5, 6$, it will never leave this class and will have the limit distribution in this class,

$$\pi^{lim} = (0, 0, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}).$$

Also if the Markov chain starts in the transient class, ie if $X_0 = 1, 2$, it will leave the transient class for the absorbing class at some point and again will have the limit distribution in the absorbing class

$$\pi^{lim} = (0, 0, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}).$$

(b) [3pt] What is the expected number of steps it takes to reach state 5 from state 4?

Solution Let $T_5 := \min\{n \geq 0 \colon X_n = 5\}$ and $m_i := E(T_5 \mid X_0 = i)$. The question is what is m_4.

By conditioning on the 1st step we get the system

$$\begin{cases} m_4 = 1 + 1/4m_4 + 1/4m_6, \\ m_6 = 1 + 1/4m_4 + 1/4m_6. \end{cases}$$

The RHS for m_4 is the same as the RHS for m_6, ie $m_4 = m_6$. We plug $m_4 = m_6$ into the 1st equation and get

$$\begin{aligned} &m_4 = 1 + 1/4m_4 + 1/4m_4, \\ &m_4 = 2. \end{aligned}$$

(c) [3pt] What is the probability that it takes *no more than* two steps to reach state 5 from state 4?

Solution 1 We have

$$\begin{aligned} &P(T_5 \leq 2 \mid X_0 = 4) \\ &= P(X_1 = 5 \mid X_0 = 4) + P(X_2 = 5, X_1 = 4 \mid X_0 = 4) + P(X_2 = 5, X_1 = 6 \mid X_0 = 4) \\ &= p_{45} + p_{44}p_{45} + p_{46}p_{65} = 1/2 + 1/4 * 1/2 + 1/4 * 1/2 = 3/4. \end{aligned}$$

Solution 2 We consider a new Markov chain Y_n where state 5 is absorbing, ie the transition matrix is

$$\tilde{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Making state 5 absorbing does not affect the time to reach 5 from 4, however now that state 5 is absorbing, we can say that reaching state 5 *before or at time 2* is equivalent to being in state 5 *at time 2*, and hence

$$P(T_5 \leq 2 \mid X_0 = 4) = P(Y_2 = 5 | Y_0 = 4) = (\tilde{P}^2)_{45} = \text{row } 4 * \text{column } 5 = 3/4.$$

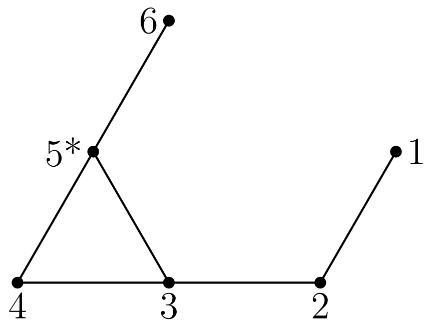
(d) [**4pt**] What is the probability that it takes *more than* ten steps to reach state 5 from state 4? An analytic-form answer suffices.

Solution Following Solution 2 of the previous question, now that state 5 is absorbing, we can say that reaching state 5 only after time 10 is equivalent to not being at state 5 at time 10, and hence

$$P(T_5 > 10 \mid X_0 = 4) = P(Y_{10} \neq 5 | Y_0 = 4) = 1 - (\tilde{P}^{10})_{45}.$$

Question 3. You navigate the city by an electric rental car following the map below. The nodes 1, ..., 6 are the parking lots where you can pick the car at the start of a rental or leave the car at the end of a rental. Having started your rental at a specific location, you end the rental and park the car at one of the *neighbouring* locations, either of them equally likely. For the next rental you pick the car up from the location where you left it last time.

To clarify: neighbouring locations are those directly connected by an edge, e.g. location 4 has two neighbours: 3 and 5*; location 3 has three neighbours: 2, 4, and 5*.



For each rental you are charged a fixed amount €c. In addition, you are charged extra or get a bonus depending on where you park the car at the end of the rental. Location 5 is also a charging station. When you leave the car at location 5, you get a bonus of €d. When you leave the car at any other location, you are charged proportionally to the *distance* to location 5, €k per unit of distance.

To clarify: the *distance* between two locations, in units, is the length of the shortest path, in edges. E.g. the distance between locations 2 and 5* is 2 units, corresponding to the path 2-3-5* (not 3 units corresponding to the path 2-3-4-5*).

(a) [4pt] Formulate a discrete-time Markov chain that is suitable to analyse the long-run average costs per rental.

Solution Let X_n = location where the car is parked at the end of rental n. The *state space* is $S = \{1, 2, 3, 4, 5, 6\}$ and the *transition matrix* is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The *Markov property* follows implicitly from the formulation: if we start the rental at a certain location, ie we parked the car there last time, then *no matter what has happened before*, next time we will park the car in one of the neighbouring locations. Eg

explicitly given: $P(X_{n+1} = 4 \mid X_n = 3) = 1/3$,
implicitly also meant: $P(X_{n+1} = 4 \mid X_n = 3, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = 1/3$
for any $i_{n-1}, \ldots, i_0$.

The transition probabilities do not depend on n, eg $P(X_{n+1} = 4 \mid X_n = 3) = 1/3$ for all n. *To summarize, X_n, $n \geq 0$, is a time-homogeneous DTMC.* We will use this MC in (b) to analyse the time-average costs.

(b) [7pt] Calculate the average costs per rental in the long-run.
Hint: from the balance equations for π_i, $i = 1, \ldots, 4$, it follows that

$$\pi_2 = 2\pi_1, \quad \pi_3 = 3\pi_1, \quad \pi_4 = 2\pi_1;$$

you can use these relations without derivation.

Solution Since we have a finite state space and an irreducible Markov chain, then we will be able to solve the balance and normalization equations and the solution will give us the *occupancy distribution* for any initial state $X_0 = i$ / initial distribution $\pi^{(0)}$.

This is the system of balance and normalization equations:

$$
\begin{aligned}
&\pi_1 = \pi_2 * 1/2,\\
&\pi_2 = \pi_1 + \pi_3 * 1/3,\\
&\pi_3 = \pi_2 * 1/2 + \pi_4 * 1/2 + \pi_5 * 1/3,\\
(4)\quad &\pi_4 = \pi_3 * 1/3 + \pi_5 * 1/3,\\
&\pi_5 = \pi_3 * 1/3 + \pi_4 * 1/2 + \pi_6,\\
(6)\quad &\pi_6 = \pi_5 * 1/3,\\
&\sum_{i=1}^{6} \pi_i = 6.
\end{aligned}
$$

As we plug in the hint

$$\pi_2 = 2\pi_1, \quad \pi_3 = 3\pi_1, \quad \pi_4 = 2\pi_1,$$

into (4), it follows that

$$\pi_5 = 3\pi_1,$$

and then from (6) it follows that

$$\pi_6 = \pi_1.$$

The blue equations give everything in terms of π_1, we plug them into the normalization equation and get the solution

$$\pi^{occ} = (\tfrac{1}{12}, \tfrac{2}{12}, \tfrac{3}{12}, \tfrac{2}{12}, \tfrac{3}{12}, \tfrac{1}{12}).$$

Finally, the average costs per rental in the long-run are, in €,

$$c - \pi_5^{occ} * d + \pi_1^{occ} * 3k + \pi_2^{occ} * 2k + \pi_3^{occ} * k + \pi_4^{occ} * k + \pi_6^{occ} * k = c - \tfrac{3}{12} d + \tfrac{13}{12} k,$$

where we used the fact that the distances to location 5* are

from location	1	2	3	4	6
distance to 5*	3	2	1	1	1

Question 4. A service desk serves two types of customers, A and B, which arrive according to independent Poisson processes with rates $\lambda_A = 1$ per hour and $\lambda_B = 3$ per hour. Each customer, of type A or B, independently of the others, will require a follow-up service with probability p.

(a) [2pt] What is the probability that it takes longer than 20 minutes from one arrival of type B till the next arrival of type B?

Solution Since inter-arrival times in the arrival process of type B customers are exponentially distributed with rate $\lambda_B = 3$ per hour and since 20 mins = 1/3 hour, the answer is

$$P(\mathrm{Exp}(3) > 1/3) = e^{-3*1/3} = e^{-1}.$$

(b) [5pt] What is the joint probability that the following happens during the first working hour: in the first half-hour at most the expected number of customers arrive and in the second half-hour no customers arrive that will need a follow-up service?
Reminder: For a random variable $X \sim \text{Poisson}(\lambda)$, $EX = \lambda$.

Solution Let
- $N_A(t), t \geq 0$ denote the arrival process of type A customers,
- $N_B(t), t \geq 0$ denote the arrival process of type A customers,
- $N(t) := N_A(t) + N_B(t), t \geq 0$, be the total arrival process of all customers,
- $N_{\text{follow-up}}(t), t \geq 0$, denote the arrival process of customers of type A and type B who require a follow-up service.

The question is what is

$$(*) \quad P\big(N(1/2) \leq E\big(N(1/2)\big), N_{\text{follow-up}}(1) - N_{\text{follow-up}}(1/2) = 0\big)?$$

We note the following:
(1) $N(\cdot) \sim PP(\lambda_A + \lambda_B)$ as the merger of the two independent Poisson processes $N_A(\cdot) \sim PP(\lambda_A)$ and $N_B(\cdot) \sim PP(\lambda_B)$;
(1') in particular, $N(1/2) \sim Poi\big((\lambda_A+\lambda_B)*1/2\big) = Poi\big(2\big)$ and, by the hint, $E\big(N(1/2)\big) = 2$;
(2) $N_{\text{follow-up}}(\cdot)$ is a probability p thinning of the Poisson process $N(\cdot)$, and hence $N_{\text{follow-up}}(\cdot) \sim PP\big((\lambda_A + \lambda_B)p\big)$;
(2') $N_{\text{follow-up}}(1) - N_{\text{follow-up}}(1/2) \sim Poi\big((\lambda_A + \lambda_B)p * (1 - 1/2)\big) = Poi\big(2p\big)$;
(3) the two events in (*) concern the non-overlapping intervals $(0, 1/2]$ and

(1/2, 1] and hence are independent.

We can now answer the question,

$$
\begin{aligned}
&P\big(N(1/2) \le E\big(N(1/2)\big), N_{\text{follow-up}}(1) - N_{\text{follow-up}}(1/2) = 0\big) \\
&\overset{(3)}{=} P(N(1/2) \le E\big(N(1/2)\big)) * P\big(N_{\text{follow-up}}(1) - N_{\text{follow-up}}(1/2) = 0\big) \\
&\overset{(1'),(2')}{=} P(Poi(2) \le 2)) * P\big(Poi(2p) = 0\big) \\
&= e^{-2}(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}) * e^{-2p}\frac{(2p)^0}{0!} = 5e^{-2-2p}.
\end{aligned}
$$

(c) [5pt] What is the probability that the second customer of type A arrives before the second customer of type B?

Solution Denote by A_i and B_i, $i \ge 1$, the inter-arrival times for, respectively, type A and type B customers. Also we introduce a notation for the event of interest,

$$E := \{\text{2nd type } A \text{ customer arrives before 2nd type } B \text{ customer}\}.$$

There are three possible scenarios for the event E to happen:

- $E_1 := \{A_1 \text{ wins from } B_1,\ A_2 \text{ wins from remaining } B_1\}$,
- $E_2 := \{A_1 \text{ wins from } B_1,\ A_2 \text{ loses to remaining } B_1, \text{ remaining } A_2 \text{ wins from } B_2\}$,
- $E_3 := \{A_1 \text{ loses to } B_1, \text{ remaining } A_1 \text{ wins from } B_2,\ A_2 \text{ wins from remaining } B_2\}$.

The inter-arrival times A_i are $\text{Exp}(\lambda_A)$ and due to the memoryless property, the remaining inter-arrival times A_i are $\text{Exp}(\lambda_A)$ as well. Similarly, the inter-arrival and remaining inter-arrival times B_i are $\text{Exp}(\lambda_B)$. Also there is independence between the pairs of competing exponentials in E_1, E_2, E_3. Hence, we have

$$
\begin{aligned}
P(E) &= P(E_1) + P(E_2) + P(E_3) \\
&= \frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} + \frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_B}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} + \frac{\lambda_B}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B}\frac{\lambda_A}{\lambda_A + \lambda_B} \\
&= \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{3}{4}\frac{1}{4} + \frac{3}{4}\frac{1}{4}\frac{1}{4} = \frac{5}{32}.
\end{aligned}
$$