Resit exam Stochastic Modelling (X_400646)

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Faculty of Science

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This exam consists of five exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. Please provide a clear and brief motivation of all your answers. Good luck!

1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) [3 pt] Draw the transition diagram of this Markov chain, determine and classify the communicating classes, and determine the period of each state.

(b) [5 pt] Suppose the Markov chain starts in state 1. Does it have a limiting distribution and/or occupancy distribution? If so, determine this limiting and/or occupancy distribution.

2. A small boarding-house provides lodgings for at most three guests at a time on a monthly basis. Suppose that on the 1st of January, three guests are staying in the boarding house. Every month, each guests independently decides to either stay for another month with probability 1/2, or leave. Assume for the moment that no new lodgers come to the boarding-house at all.

(a) [5 pt] Model the number of guests staying in the boarding-house at the start of each month as a discrete-time Markov chain, and determine the expected number of months it takes until all guests have left.

(b) [2 pt] Now suppose that every month, a potential new lodger arrives with probability p (with probability $1 - p$, no person seeking lodgings arrives). If the boarding-house has rooms available, the new lodger will become a guest starting from the 1st day of the *next* month. Model this situation using a discrete-time Markov chain, and draw its transition diagram.

3. Consider the following queueing system: customers arrive according to a Poisson process with rate 1. A fraction 2/3 of these customers are *slow* and require a service time that follows an exponential distribution with rate 1. The other customers are *fast*: their service time has an exponential distribution with rate $\alpha > 1$. There is a single server. Time is measured in minutes.

(a) [4 pt.] What is the joint probability that between 9:00 and 9:10, at least three fast customers arrive but no customers arrive at all before 9:02 ?

(b) [4 pt.] Suppose that 15% of the slow customers require special attention. What is the probability that after 10:00, exactly two slow customers requiring special attention arrive before the first fast customer appears?

(c) [6 pt.] Show that the expected waiting time of a customer in equilibrium is given by

$$E(W^q) = \frac{2\alpha^2 + 1}{\alpha^2 - \alpha} \qquad (\alpha > 1).$$

(d) [3 pt.] Make a sketch of the graph of $E(W^q)$ as a function of α and explain its behaviour. In particular, can you explain what happens to $E(W^q)$ in the limits $\alpha \to 1^+$ and $\alpha \to \infty$?

4. A company is considering building a new storage facility where customers can rent a storage unit for a longer period of time. The company needs to decide on the number of storage units to build at the new facility. It expects about 88 new customers per year according to a Poisson process, who will each rent a storage unit for an average period of 2 years.

(a) [2 pt.] If it were possible to provide an infinite number of storage units, what would be the long-run distribution of the number of rented units?

(b) [3 pt.] Suppose that it is most cost-effective to construct a storage facility with a number of storage units that is a multiple of 20. How many storage units would you recommend the company to build, and why? What are your objections against providing a larger or smaller number of storage units?

5. A service station has a waiting area for two customers. Customers arrive according to a Poisson process with rate λ. If the waiting area is full when a customer arrives, he goes elsewhere. There are two servers. As long as the total number of customers in the service station is at most two, one server helps the customers and the other one works on other tasks. If there are more than two customers present, both servers help the customers. We assume that all service times follow an exponential distribution with rate μ.

(a) [4 pt.] Model the system as a continuous-time Markov chain, draw its transition rate diagram, and formulate the balance equations.

(b) [4 pt.] Suppose that $\lambda = \mu$. Determine the long-run fraction of time it is the case that both servers are busy helping customers in this special case.

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Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khintchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$
$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$
$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$
$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0, 1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.