Solutions Resit exam Stochastic Modelling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

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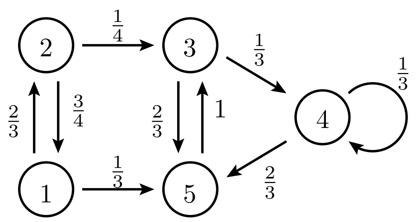
1.

(a)

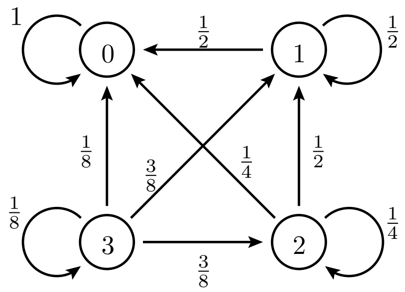


Communicating classes: $\{1,2\}$ transient, $\{3,4,5\}$ absorbing. States $1,2$ have period 2, states $3,4,5$ have period 1.

(b) There is one absorbing class, which is aperiodic, so there is a unique invariant distribution which is the limiting and occupancy distribution. Solving the balance equations, we obtain that it is given by $\pi = \left(0, 0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$.

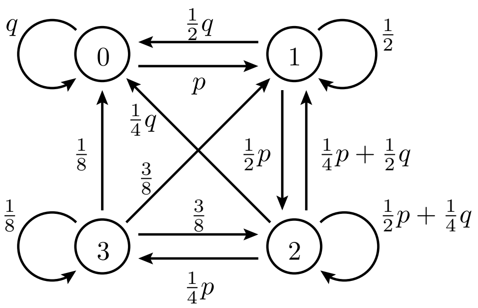
2.

(a) Let X_n be the number of guests staying in the boarding-house at the start of the n-th month (month 0 is January). The transition diagram is



Define $m_i = E(T_0 \mid X_0 = i)$. By conditioning on the first step, we obtain a system of equations for m_1, m_2, m_3, from which we obtain $m_3 = 22/7$.

(b) With $q = 1 - p$, the new transition diagram is



3.

(a) The total number of customers arriving between 9:00 and 9:02 follows a Poisson distribution with parameter 2, and by Poisson splitting, the number of fast customers arriving between 9:02 and 9:10 has a Poisson distribution with parameter 8/3. Moreover, these numbers are independent. So the desired probability is

$$e^{-2} \times \left(1 - \left(1 + \frac{8}{3} + \frac{(8/3)^2}{2!}\right)\right) e^{-8/3} = e^{-2} - \frac{65}{9}\, e^{-2\frac{8}{3}}.$$

(b) Let T_s be the waiting time for a slow customer requiring special attention, and let T_f be the waiting time for a fast customer. Then, by Poisson splitting, $T_s \sim \text{Exp}(1/10)$ and $T_f \sim \text{Exp}(1/3)$. Since these waiting times are exponential and memoryless, the desired probability is

$$P(T_s < T_f)^2 \cdot P(T_f < T_s) = \left(\frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{3}}\right)^2 \cdot \left(\frac{\frac{1}{3}}{\frac{1}{10} + \frac{1}{3}}\right) = \left(\frac{3}{13}\right)^2 \cdot \frac{10}{13}.$$

(c) The system is M/G/1. Customers are slow with probability 2/3 and service time $B_s \sim \text{Exp}(1)$, and fast with probability 1/3 and service time $B_f \sim \text{Exp}(\alpha)$. It follows that

$$E(B) = \frac{1 + 2\alpha}{3\alpha} \qquad \text{and} \qquad E(B^2) = \frac{2 + 4\alpha^2}{3\alpha^2},$$

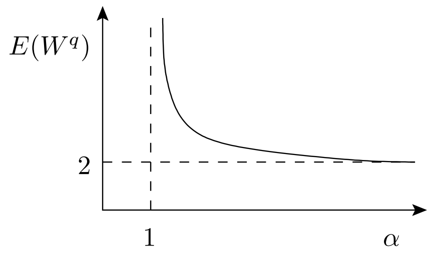
hence

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{1 + 2\alpha^2}{\alpha + 2\alpha^2}.$$

So since $\rho = \lambda\, E(B) = E(B)$,

$$E(W^q) = \frac{\rho}{1 - \rho}\, E(R) = \frac{2\alpha^2 + 1}{\alpha^2 - \alpha}.$$

(d)



For $\alpha \to 1^+$, the service times of all customers become Exp(1), so $\rho \to 1$, hence $E(W^q) \to \infty$. For $\alpha \to \infty$, the service times of fast customers vanish, so we end up with an M/M/1 system of slow customers with arrival rate 2/3 and $B \sim$ Exp(1), and hence $E(W^q) \to 2$.

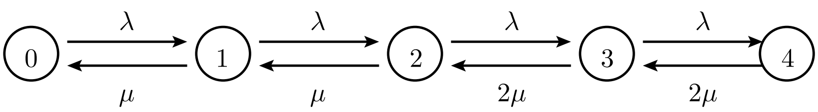
4.

(a) This is an M/G/∞ system with $\rho = \lambda E(B) = 2 \cdot 88 = 176$, so the limiting distribution is Poisson(176).

(b) The limiting distribution in (a) is approximately normal with mean 176 and standard deviation $\sqrt{176} \approx 13.3$. 220 units or more is more than 3 standard deviations above the mean, which means we could rent out that many units only a tiny fraction of the time. 200 units is about 1.8 standard deviations above the mean, while 180 is only a little bit above the mean. This means we can expect to be able to rent out between 180 and 200 units a reasonably large fraction of the time, hence our best choice is to build 200 units. With 180 units or less, we run the risk of losing too many customers.

5.

(a) Let $X(t)$ be the number of customers in the system at time t.



Balance equations:

$$\lambda p_0 = \mu p_1 \quad \lambda p_1 = \mu p_2 \quad \lambda p_2 = 2\mu p_3 \quad \lambda p_3 = 2\mu p_4$$

(b) For $\lambda = \mu$, the normalized solution of the balance equations is

$$(p_0, p_1, p_2, p_3, p_4) = (\tfrac{4}{15}, \tfrac{4}{15}, \tfrac{4}{15}, \tfrac{2}{15}, \tfrac{1}{15}),$$

so the fraction of time both servers are busy is $p_3 + p_4 = 1/5$.