Final exam Stochastic Modeling (X_400646)

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This exam consists of three exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. Please provide a clear and brief motivation of all your answers. Good luck!

1. A fundamental result in the study of queueing systems is Little's law.

(a) [6 pt.] For a stable queueing system in equilibrium, formulate Little's law for the relation between $E(L^q)$, the mean number of customers in the queue, and $E(W^q)$, the mean waiting time, and provide a clear argument explaining why it must hold.

(b) [6 pt.] For a stable M/M/1 queue, formulate the arrival relation between $E(L^q)$ and $E(W^q)$, explain clearly why it must hold, and combine it with Little's law to determine $E(L^q)$ and $E(W^q)$.

2. A small supermarket with a single checkout is experimenting with a self-service checkout machine with facial recognition. At the checkout, customers scan the items they want to buy, and leave; the amount due is deducted from their bank account automatically. We assume that it takes a customer exactly one unit of time to scan each item, that the number of items scanned by each customer follows a Poisson distribution with mean 5, and that customers arrive at the checkout according to a Poisson process with rate λ.

(a) [3 pt.] Which queueing model describes the number of customers at the checkout of the supermarket, and for which values of λ is the model stable?

(b) [6 pt.] The supermarket is interested in how the expected waiting time of customers at the checkout depends on how busy it is. Give a formula for $E(W^q)$, and determine $E(W^q)$ when the self-service machine is in use 20% of the time, 40% of the time, 60% of the time, and 80% of the time.

(c) [6 pt.] Consider the customers who arrive at the checkout while there is exactly one other customer present (who is using the self-service machine). For these customers, what is the probability that their waiting time at the checkout is at most two units of time?
(Beware that the service time has a discrete distribution.)

3. Consider a queueing system with a single queue and two servers: a fast server, and a slow server. Customers arrive according to a Poisson process with a rate of 1 per minute and are served in FCFS order. Service times at both servers follow an exponential distribution, but while the mean service time is 1 minute for the fast server, it is 2 minutes for the slow server. Customers always go to the fast server if they have a choice between the two servers, and otherwise go to the first server that is available.

To model this queueing system we use a continuous-time Markov chain on the state space $S = \{0, 1_f, 1_s, 2, 3, 4, \dots\}$, where the states $0, 2, 3, 4, \dots$ stand for the total number of customers in the system, and states 1_f and 1_s indicate that there is one customer in the system who is served by the fast or the slow server, respectively.

(a) [4 pt.] Draw the transition diagram for this Markov chain.

The equilibrium probability that the system is in state 2 turns out to be $p_2 = 3/17$ (you do not have to show this).

(b) [6 pt.] Use balance equations and $p_2 = 3/17$ to determine the equilibrium probabilities of the states 0, 1_f and 1_s, and use this to calculate the fraction of time the slow server is busy, and the fraction of time the fast server is busy.

(c) [8 pt.] Use balance equations and $p_2 = 3/17$ to determine the equilibrium probabilities p_{2+i} of the states $2 + i$ for $i = 0, 1, 2, \dots$, and use this to show that the equilibrium waiting time of a customer in the queue has density

$$f_{W^q}(t) = \frac{9}{34}\, e^{-\frac{1}{2}t} \qquad (t > 0).$$

(Note: you can use your answers in (b) and (c) to verify that $p_2 = 3/17$.)

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Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \leq x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khintchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$
$$E(BP) = \frac{E(B)}{1-\rho}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$
$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$
$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)}\, e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0,1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.