Solution Final exam Stochastic Modeling (X_400646)

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1.

(a) Little's law says that

$$E(L^q) = \lambda E(W^q),$$

where λ is the (long-run) rate at which customers arrive. If customers have to pay a dollar for every unit of time they spend waiting in the queue, then each side of Little's law represents the long-run revenue earned by the system per unit time (see the course literature for details).

(b) The arrival relation for M/M/1 is

$$E(W^q) = \frac{1}{\mu}E(L^q) + \frac{\rho}{\mu},$$

where μ is the service rate, and $\rho = \lambda/\mu$ is the fraction of time the server is busy. It holds because an arriving customer has to wait for i) the service times of customers who are already in the queue, and ii) the residual service time of the customer in service, if the server is busy. Combining the arrival relation with Little gives

$$E(W^q) = \frac{1}{\mu}\frac{\rho}{1-\rho} \qquad E(L^q) = \frac{\rho^2}{1-\rho}\,.$$

2.

(a) This is an M/G/1 system with $B \sim \text{Poi}(5)$. We have $\rho = \lambda\, E(B) = 5\lambda$, so the system is stable for all $\lambda < 1/5$.

(b) We have $E(B) = \text{Var}(B) = 5$, so $c_B^2 = 1/5$, and by Pollaczek–Khintchine

$$E(W^q) = \frac{1}{2}\frac{\rho}{1-\rho}\,(1 + c_B^2)\,E(B) = \frac{3\rho}{1-\rho}.$$

Hence

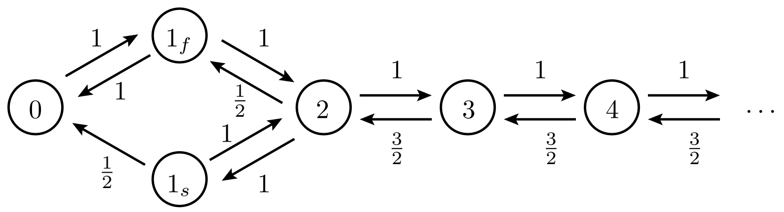
ρ	20%	40%	60%	80%
$E(W^q)$	3/4	2	9/2	12

(c) The waiting time for these customers is the residual service time R of the customer in service, so

$$\begin{aligned} P(R \le 2) &= \frac{1}{E(B)} \int_0^2 P(B > u)\, du \\ &= \frac{1}{5}\left(\int_0^1 \left(1 - e^{-5}\right) du + \int_1^2 \left(1 - e^{-5} - 5e^{-5}\right) du \right) \\ &= \frac{1}{5}\left(2 - 7e^{-5}\right). \end{aligned}$$

3.

(a)



(b) The balance equations for the states $\{0, 1_f, 1_s, 2\}$ are

$$p_0 = p_{1_f} + \tfrac{1}{2}\, p_{1_s} \qquad 2\, p_{1_f} = p_0 + \tfrac{1}{2}\, p_2 \qquad \tfrac{3}{2}\, p_{1_s} = p_2$$

where $p_2 = 3/17$. This has the solution $p_0 = 7/34$, $p_{1_s} = 2/17$, $p_{1_f} = 5/34$. The fractions of time that the fast and slow server are busy are, respectively, $1 - p_0 - p_{1_s} = 23/34$ and $1 - p_0 - p_{1_f} = 22/34$.

(c) "Global balance" gives for the states $2 + i$, $i \ge 0$,

$$p_{2+i} = \tfrac{3}{2}\, p_{2+i+1} \qquad \Longrightarrow \qquad p_{2+i} = \left(\tfrac{2}{3}\right)^i p_2.$$

If an arriving customers finds $2 + i$ $(i \ge 0)$ customers already in the system, his waiting time follows an Erlang$(i + 1, \frac{3}{2})$ distribution; by PASTA, the probability that this happens is p_{2+i}. Therefore, for $t > 0$,

$$f_{W^q}(t) = \sum_{i=0}^{\infty} \tfrac{3}{2}\, e^{-\frac{3}{2}t} \frac{\left(\frac{3}{2}t\right)^i}{i!}\, p_{2+i} = \tfrac{3}{2}\, p_2\, e^{-\frac{3}{2}t} \sum_{i=0}^{\infty} \frac{t^i}{i!} = \frac{9}{34}\, e^{-\frac{1}{2}t}.$$