Midterm exam Stochastic Modelling (X_400646)

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October 21, 2020, 12:15–14:15

This exam consists of five exercises, for which you can obtain 45 points in total. Your grade will be calculated as (number of points + 5)/5. Please provide a clear and brief motivation of all your answers. Good luck!

1. Consider a discrete-time Markov chain on the state space $\{1, 2, 3, 4, 5\}$ with transition matrix

$$\mathrm{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

(a) [2 pt] Draw the transition diagram of this Markov chain.

(b) [8 pt] Determine the limiting distribution of the Markov chain if it starts in state 1, i.e., when $P(X_0 = 1) = 1$.

2. An investor starts with an investment capital of 40 k€ and aims to double his capital to 80 k€ by investing in start-up businesses. He invests 20 k€ in a start-up at a time. If the start-up is not succesful, he loses his investment, but if the start-up is successful, the investor receives back his investment and makes an additional profit of 20 k€ on top. The probability that a start-up is successful is 2/3.

(a) [3 pt] Formule a discrete-time Markov chain to model the investor's capital after each investment, and draw its transition diagram.

(b) [6 pt] Calculate the probability that the investor manages to reach his target capital of 80 k€ before losing all his money.

3. At a bus stand, Number 2 buses arrive according to a Poisson process with a rate of 2 buses per hour. Number 7 buses arrive according to an independent Poisson process with a rate of 7 buses per hour.

(a) [4 pt] What is the probability that at least two buses pass by in half an hour? Do not forget to explain your answer.

(b) [4 pt] You want to take the first Number 2 bus that arrives. What is the probability that exactly three Number 7 buses pass by while you are waiting for a Number 2 bus? Explain your answer.

4. [8 pt] Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain on a countable state space S with transition matrices $\mathrm{P}(t)$. Give a precise definition of the *transient distribution* $p(t)$ at time t of the Markov chain, and show that

$$p(s+t) = p(s)\,\mathrm{P}(t) \qquad \text{for } s, t \geq 0.$$

5. A small shuttle bus service between two locations A and B operates as follows. At each location, one shuttle bus waits for passengers. As soon as one of the two buses has two passengers, *both* buses drive to the other location, unload their passengers, and the process is repeated. We assume that passengers arrive at each location according to a Poisson process with rate λ, and that the travel time between the two locations is negligible (compared to the inter-arrival times of passengers) and can therefore be ignored.

(a) [4 pt] Formulate a continuous-time Markov chain to describe the operation of the shuttle service. State clearly what each state of the Markov chain means, and draw the transition rate diagram.

(b) [6 pt] Determine the long-run expected number of passengers who are waiting for a shuttle bus to depart.