**Solutions Midterm exam Stochastic Modelling (X_400646)**

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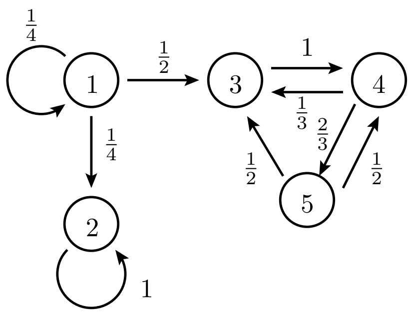
**1**.

(a)

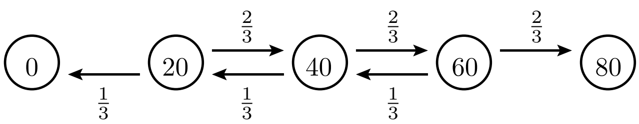



(b) There are two absorbing communicating classes: $\{2\}$ and $\{3,4,5\}$. Each class is aperiodic. The invariant distribution associated with class $\{2\}$ is $\pi = (0,1,0,0,0)$, while the invariant distribution associated with class $\{3,4,5\}$ is $\pi = \left(0,0,\frac{2}{7},\frac{3}{7},\frac{2}{7}\right)$. If the Markov chain starts in state 1, it eventually ends up in the class $\{2\}$ with probability $\frac{1}{3}$, and in class $\{3,4,5\}$ with probability $\frac{2}{3}$. Hence, the limiting distribution in this case is

$$\pi = \tfrac{1}{3}(0,1,0,0,0) + \tfrac{2}{3}\left(0,0,\tfrac{2}{7},\tfrac{3}{7},\tfrac{2}{7}\right) = \left(0,\tfrac{1}{3},\tfrac{4}{21},\tfrac{2}{7},\tfrac{4}{21}\right).$$

**2**.

(a) Let $X_n$ be the investor's capital (in k€) after the $n$-th investment (with $X_0$ the starting capital).




(b) Define $q_i = P(\,T_{80} < T_0 \mid X_0 = i\,)$, where $T_N$ is the first-passage time to state $N$. We want to know $q_{40}$. Conditioning on the first step leads to a system of equations for $q_{20}$, $q_{40}$ and $q_{60}$, from which we obtain $q_{40} = 4/5$.

**3.**

(a) By Poisson merging, buses arrive at the bus stand according to a Poisson process with a rate of 9 buses per hour. Hence, the number of buses passing by in half an hour has a Poisson distribution with parameter 9/2. The probability that at least two buses pass by in half an hour is therefore

$$1 - e^{-9/2} - \tfrac{9}{2}\,e^{-9/2}.$$

(b) The waiting time for a Number $i$ bus is $T_i \sim \text{Exp}(i)$, where $i \in \{2, 7\}$. The waiting times have the lack of memory property, so the probability that exactly three Number 7 buses pass by while you are waiting for a Number 2 bus is

$$P(\,T_7 < T_2\,)^3 \times P(\,T_2 < T_7\,) = \big(\tfrac{7}{9}\big)^3 \times \tfrac{2}{9}.$$

**4.** The *transient distribution* $p(t)$ at time $t$ is the row vector with components $p_j(t) = P(\,X(t) = j\,)$, $j \in S$. By conditioning on the state at time $s$, we get $p_j(s+t) = \sum_{i\in S} p_i(s)\, p_{ij}(t)$ for each $j \in S$, and hence $p(s+t) = p(s)\,\mathrm{P}(t)$.

**5.**

(a) Option 1 is to take states $(a, b)$ where $a$ and $b$ are the numbers of passengers waiting at locations A and B, respectively. Option 2 is to take states $n \in \{0, 1, 2\}$ where $n$ is the total number of passengers waiting for their shuttle bus to leave.


(b) For Option 1, the invariant distribution is

$$p = \big(\,p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}\,\big) = \big(\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\big).$$

For Option 2, the invariant distribution is

$$p = (\,p_0, p_1, p_2\,) = \big(\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}\big).$$

Hence, the long-run expected number of waiting passengers is equal to 4/5.