Resit exam Stochastic Modeling (X_400646)

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This exam consists of five exercises, for which you can obtain 45 points in total (the distribution of points is indicated between square brackets). Your grade will be calculated as (number of points + 5)/5. Please write your name and student number on every paper you hand in, and give a clear and brief motivation of all your answers. A formula sheet is included on the last page of this exam. The use of books or a graphical calculator during the exam is not allowed. Good luck!

1. Consider a store that keeps a certain product on its shelves for which the demand is stable over time. The number of products on the shelves (the stock) takes values in the set $S = \{0, 1, 2, 3\}$. If the stock at the end of the day is less than 2, the store places an order that is delivered early the next morning, before the store opens. Let X_n denote the stock at the end of day n. It turns out that $\{X_n, n = 0, 1, 2, \dots\}$ is a discrete-time Markov chain with state space S and transition matrix

$$\mathrm{P} = \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \end{pmatrix}$$

(a) [2 pt.] Draw the transition diagram for this discrete-time Markov chain.

(b) [3 pt.] Determine the probability that the daily demand for the product is 0, and do the same for the probabilities that the daily demand is 1, 2 or 3. Also, determine how many products are ordered by the store when the stock at the end of the day is 0 or 1.

(c) [5 pt.] Determine the long-run fraction of days on which the product is out of stock at the end of the day.

(d) [5 pt.] Suppose that at the end of a given day, the store has 3 items of the product in stock. Calculate the expected number of days it takes until the product is out of stock at the end of the day for the first time.

2. [5 pt.] Show that if $\pi^* = (\pi_j^*)$ is a stationary distribution of a discrete-time Markov chain with state space S, then it satisfies the balance equations

$$\pi_j^* = \sum_{k \in S} \pi_k^* p_{kj} \qquad (j \in S)$$

3. Consider a checkout in a small supermarket. Customers arrive according to a Poisson process with an average of 10 customers per hour. There are two types of customer: the service time of type 1 customers follows a uniform distribution between 1 and 3 minutes, and the service time of type 2 customers has an exponential distribution with a mean of 1.5 minutes. The fraction of customers who are of type 1 is given by p.

(a) [5 pt.] Calculate the joint probability that the following two events occur together: (i) in the next 6 minutes, exactly two customers of type 1 arrive, and (ii) the first customer to arrive after those 6 minutes is of type 2.

(b) [5 pt.] Suppose a manager comes in at an arbitrary moment in time, and the cashier is busy serving a customer. In the special case $p = 0$ (all customers of type 2), what is the probability that it will take at least 2 minutes until the cashier has finished serving the customer? Answer the same question in the case $p = 1$ (all customers of type 1).

4. [5 pt.] For a stable $M/G/1$ queue operating under the LCFS-NP service discipline, formulate the arrival relation for the expected waiting time $E(W^q)$ in terms of the load ρ, the expected residual service time $E(R)$, and the expected busy period $E(BP)$, and use this to show that

$$E(W^q) = \frac{\rho}{1-\rho} E(R)$$

5. Consider the following queueing model. Customers arrive according to a Poisson process with rate λ. A customer always goes into service if there are no other customers present when he arrives. However, if upon arrival there are already $k \geq 1$ customers present in the system, then the customer joins the queue with probability $1/k$, or leaves (and is lost) otherwise. The service times follow an exponential distribution with parameter μ.

(a) [3 pt.] Draw the transition diagram of the continuous time Markov chain that describes the number of customers in the system as a function of time.

(b) [2 pt.] For which values of λ and μ is this sytem stable? Please explain your answer.

(c) [5 pt.] Formulate the balance equations for this system, and from them derive the limiting distribution for the number of customers in the system.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khintchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0, 1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$