**Solutions Resit exam Stochastic Modeling**

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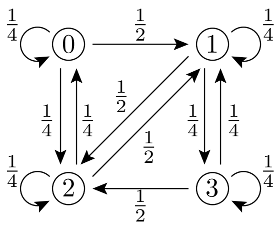
**1.**

(a)




(b) Considering the possible transitions from states 2 and 3, we see that the demand is 0 with probability 1/4, 1 with probability 1/2, and 2 with probability 1/4. It follows that the store apparently orders 2 units of the product when the stock is 0 or 1 at the end of the day.

(c) Let $\pi_i$ denote the long-run fraction of days that the DTMC spends in state $i$. These $\pi_i$ must satisfy the balance equations

$$\pi_0 = \frac{1}{4}\pi_0 + \frac{1}{4}\pi_2 \qquad \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3$$
$$\pi_3 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_3 \qquad \pi_2 = \frac{1}{4}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3$$

The two equations on the left give $\pi_2 = 3\pi_0$ and $\pi_1 = 3\pi_3$. Substituting this into the first equation on the right gives $3\pi_3 = 2\pi_0 + \pi_3$, hence $\pi_3 = \pi_0$. By normalization, we must have that $\pi_0 + \pi_1 + \pi_2 + \pi_3 = (1 + 3 + 3 + 1)\pi_0 = 1$, so it follows that the long-run fraction of days on which the product is out of stock at the end of the day equals $\pi_0 = 1/8$.

(d) Let $m_i$ denote the expected number of days until the DTMC is in state 0, when the DTMC starts in state $i$. We want to know $m_3$. By conditioning on the first step, we have

$$m_1 = 1 + \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3$$
$$m_2 = 1 + \frac{1}{2}m_1 + \frac{1}{4}m_2$$
$$m_3 = 1 + \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3$$

Clearly, $m_1 = m_3$. Substituting into the second equation gives $m_2 = \frac{4}{3} + \frac{2}{3} m_3$. The third equation then becomes $m_3 = \frac{5}{3} + \frac{5}{6} m_3$, hence $m_3 = 10$.

**2**. Suppose that $\pi^* = (\pi_j^*)$ is a stationary distribution of the DTMC $\{X_n, n = 0, 1, 2, \dots\}$. This means that if $X_0$ has the distribution $\pi^*$, then so does $X_1$ (and also $X_2, X_3, \dots$). Therefore, letting the DTMC start from the distribution $\pi^*$, we obtain that

$$\begin{aligned} \pi_j^* = P(X_1 = j) &= \sum_{k \in S} P(X_0 = k) P(X_1 = j \mid X_0 = k) \\ &= \sum_{k \in S} \pi_k^* \, p_{kj} \qquad (j \in S). \end{aligned}$$

**3**.

(a) The Poisson arrival rate of customers is $\lambda = 1/6$ per minute. By Poisson splitting, type 1 and type 2 customers arrive according to independent Poisson processes with respective rates $p\lambda$ and $(1-p)\lambda$.

So, the number of type 1 arrivals in the first 6 minutes has a Poisson distribution with parameter $6p\lambda = p$, hence event (i) has probability $\frac{p^2}{2} e^{-p}$. The arrival process after these 6 minutes is independent from whap happens in the first 6 minutes, and the probability that the next customer to arrive is of type 2 is $1 - p$. It follows that the joint probability of the two events is

$$\frac{1}{2} p^2 (1-p) e^{-p}.$$

(b) We need to find the probability that the *residual* service time $R$ is at least 2 minutes, in the two special cases $p = 0$ and $p = 1$.

In the case $p = 0$, all customers are of type 2. Since their service times have an exponential distribution with parameter $\mu = 2/3$, by the memoryless property, $P(R > 2) = e^{-2\mu} = e^{-4/3}$.

In the case $p = 1$, the service time $B$ follows a uniform distribution on $(1, 3)$, and we have

$$\begin{aligned} P(R > 2) &= \frac{1}{E(B)} \int_2^\infty P(B > u) \, du = \frac{1}{2} \int_2^3 \frac{3 - u}{2} \, du \\ &= \frac{1}{4} \left[ 3u - \frac{1}{2} u^2 \right]_2^3 = \frac{1}{8}. \end{aligned}$$

**4**. As is explained in the lectures (we omit the explanation here), the arrival relation for a stable $M/G/1$ queue under the LCFS-NP service discipline is

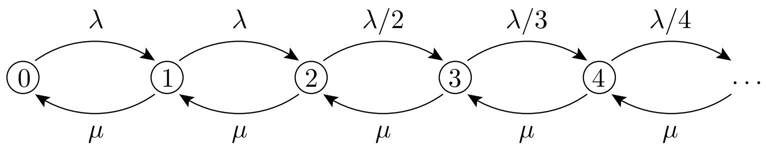
$$E(W^q) = \rho \times E(R) + \rho \times \lambda E(R) \times E(BP).$$

Using that $E(BP) = \dfrac{E(B)}{1-\rho}$ and $\lambda E(B) = \rho$, it follows that

$$E(W^q) = \frac{\rho}{1-\rho} E(R).$$

**5**.

(a)




(b) Observe that, for every fixed $k > 1$, when the number of customers in the system is at least $k$, the rate at which customers join the queue is at most $\lambda/k$, while the rate at which customers leave is $\mu$. Since for all $\lambda > 0$ and $\mu > 0$, $\mu > \lambda/k$ when $k$ is large enough, it follows that the length of the queue cannot blow up. So the system is stable for *all* positive $\lambda$ and $\mu$.

(c) The balance equations are

$$\lambda p_0 = \mu p_1$$

$$\frac{\lambda}{i-1} p_{i-1} = \mu p_i \qquad i = 2, 3, 4, \ldots$$

It follows that $p_1 = \dfrac{\lambda}{\mu} p_0$, and

$$\begin{aligned} p_i &= \frac{1}{i-1}\frac{\lambda}{\mu} p_{i-1} = \frac{1}{(i-1)(i-2)} \left(\frac{\lambda}{\mu}\right)^2 p_{i-2} \\ &= \cdots = \frac{1}{(i-1)!} \left(\frac{\lambda}{\mu}\right)^{i-1} p_1 = \frac{1}{(i-1)!} \left(\frac{\lambda}{\mu}\right)^{i} p_0 \qquad i = 2, 3, 4, \ldots \end{aligned}$$

The last equality also holds for $i = 1$. Next, we normalize these probabilities:

$$1 = \sum_{i=0}^{\infty} p_i = p_0 + \sum_{i=1}^{\infty} \frac{1}{(i-1)!} \left(\frac{\lambda}{\mu}\right)^{i} p_0 = p_0 \left(1 + \frac{\lambda}{\mu} e^{\lambda/\mu}\right).$$

So, the limiting distribution is given by

$$\begin{aligned} p_0 &= \left(1 + \frac{\lambda}{\mu} e^{\lambda/\mu}\right)^{-1} \\ p_i &= \left(1 + \frac{\lambda}{\mu} e^{\lambda/\mu}\right)^{-1} \frac{1}{(i-1)!} \left(\frac{\lambda}{\mu}\right)^{i} \qquad i = 1, 2, 3, \ldots \end{aligned}$$