Final exam Stochastic Modeling (X_400646)

Vrije Universiteit Amsterdam
Faculty of Science

December 20, 2019, 8:45–10:45

This exam consists of three exercises, for which you can obtain 36 points in total (the distribution of points is indicated between square brackets). Your grade will be calculated as (number of points + 4)/4. Please write your name and student number on every paper you hand in, and give a clear and brief motivation of all your answers. A formula sheet is included on the last page of this exam. The use of books or a graphical calculator during the exam is not allowed. Good luck!

1. For an M/M/c/c queue with arrival rate λ and service rate μ

(a) [3 pt.] draw the transition diagram;

(b) [6 pt.] derive the equilibrium distribution of the number of customers in the system (in terms of the total offered load $a = \lambda/\mu$), by formulating and solving the balance equations;

(c) [3 pt.] show that the blocking probability (i.e., the probability that an arriving customers finds all servers occupied) is given by

$$B(c, a) = \frac{a^c/c!}{\sum_{k=0}^{c} a^k/k!}.$$

2. At an art exhibit, one artist has created a room containing an interactive work of art that must be experienced by two visitors at the same time. People arrive at the room according to a Poisson process with rate λ. When there are at least two people in the queue and nobody inside the room, two people are admitted into the room together. The time they spend in the room follows an exponential distribution with parameter μ, after which the two visitors leave the room together, and the next two persons can be admitted.

(a) [3 pt.] For which values of λ and μ is the system (queue plus art room) stable? Please explain your answer. (*Hint:* how does the system behave when the number of customers in the system is very large?)

(b) [3 pt.] Draw the transition diagram of the continuous time Markov chain that describes the number of customers in the system as a function of time.

(c) [6 pt.] Suppose that per hour, on average 18 people come to visit the art room, and that a pair of visitors spends on average 4 minutes and 48 seconds in the room. If we measure time in units of two hours, this means that the parameters are $\lambda = 36$ and $\mu = 25$ (per two hours). Formulate the balance equations and verify that for the given choice of parameters, the equilibrium distribution of the total number of customers in the system is given by

$$p_0 = \frac{1}{10},$$
$$p_i = \frac{9}{50}\left(\frac{4}{5}\right)^{i-1}, \qquad i \geq 1.$$

3. In this exercise, we consider a stable M/G/1 queue with arrival rate λ, load ρ and expected service time $E(B)$.

(a) [6 pt.] The server alternately goes through busy periods (serving customers) and idle periods (no customers). Explain why the behaviour of the server can be seen as an On-off process and use this to argue that

$$E(BP) = \frac{E(B)}{1-\rho}$$

by relating the long-run fraction of time that the server is working (or the limiting probability that the server is busy) to the expected busy period $E(BP)$.

(b) [6 pt.] Consider the special case where the load is $\rho = 2/3$, and there are two types of customers: a fraction p of the customers have a service time that follows an exponential distribution with mean 2, and the remaining customers have a deterministic service time equal to 2. Calculate the expecting waiting time $E(W^q)$ and the expected number of customers in the system $E(L)$ in terms of p.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khintchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0, 1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$