Solutions Final exam Stochastic Modelling

Vrije Universiteit Amsterdam
Faculty of Science

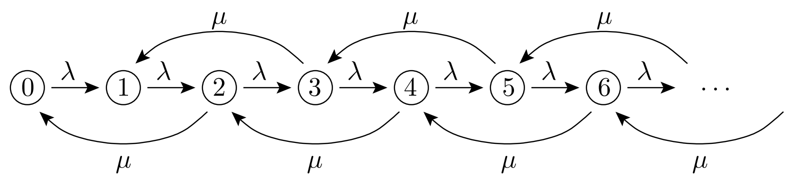
December 20, 2019

1. This exercise is about the standard M/M/c/c queueing model. Its analysis is discussed in detail in the lectures and course literature.

2.

(a) When the number of customers in the system is very large, there are constantly two visitors in the room. It follows that at rate μ we have recurring events where two customers leave the system together, so that the rate at which customers leave is 2μ. The rate at which customers arrive is λ. For the system to be stable, this arrival rate must be (strictly) smaller than the rate at which customers leave. Hence, the condition for stability is that $\lambda < 2\mu$.

(b)



(c) For $\lambda = 36$ and $\mu = 25$, the "global balance" equations are

$$\begin{cases} 36p_0 = 25p_2 \\ 36p_i = 25(p_{i+1} + p_{i+2}) \qquad i = 1, 2, 3, \dots \end{cases}$$

We first check that the proposed p_i satisfy these equations:

$$25p_2 = 25 \cdot \frac{9}{50} \cdot \frac{4}{5} = \frac{36}{10} = 36p_0$$

and, for $i \geq 1$,

$$\begin{aligned} 25(p_{i+1} + p_{i+2}) &= 25 \cdot \left(\frac{4}{5} + \frac{4^2}{5^2}\right) \cdot \frac{9}{50}\left(\frac{4}{5}\right)^{i-1} \\ &= (20 + 16) \cdot \frac{9}{50}\left(\frac{4}{5}\right)^{i-1} = 36p_i. \end{aligned}$$

(NB: alternatively, instead of checking "global balance", you can also check that the proposed solution satisfies "detailed balance", but this is more work.)

We also must check that the proposed p_i sum to 1:

$$\sum_{i=0}^{\infty} p_i = \frac{1}{10} + \frac{9}{50} \sum_{i=1}^{\infty} \left(\frac{4}{5}\right)^{i-1} = \frac{1}{10} + \frac{9}{50} \cdot \frac{1}{1-4/5} = 1.$$

We conclude that $\mathrm{p} = (p_i)$ is indeed the equilibrium distribution.

3.

(a) Define the process $\{Z(t), t \geq 0\}$ as On when the server is busy, and Off when the server is idle. Then all On periods A_i have the distribution of a busy period, and all Off periods U_i (the period between a busy period and the arrival of the next customer) follow an exponential distribution with parameter λ. Moreover, the cycles (an On period followed by an Off period) are independent. Hence, $\{Z(t), t \geq 0\}$ is indeed an On-off (renewal) process.

The fraction of time that the server is working (ρ) is also the fraction of time that the On-off process is On. It follows that

$$\rho = \frac{E(A_i)}{E(A_i) + E(U_i)} = \frac{E(BP)}{E(BP) + 1/\lambda}.$$

Solving for $E(BP)$ gives

$$E(BP) = \frac{\rho/\lambda}{1-\rho} = \frac{E(B)}{1-\rho}.$$

(b) There are two types of customers: with probability p, a customer is of type 1 and his service time has an exponential distribution with mean 2 and hence parameter $\mu = 1/2$; the remaining customers are of type 2 and have a deterministic service time equal to 2. It follows that

$$E(B) = p \cdot \frac{1}{\mu} + (1-p) \cdot 2 = 2,$$
$$E(B^2) = p \cdot \frac{2}{\mu^2} + (1-p) \cdot 2^2 = 4(1+p).$$

Hence, by the Pollaczek–Khintchine formula,

$$E(W^q) = \frac{\rho}{1-\rho} \cdot \frac{E(B^2)}{2E(B)} = \frac{2/3}{1/3} \cdot \frac{4(1+p)}{2 \cdot 2} = 2(1+p).$$

The expected sojourn time of a customer is therefore

$$E(S) = E(W^q) + E(B) = 2(2 + p).$$

From $\rho = \lambda E(B)$ it follows that the arrival rate of customers is $\lambda = 1/3$, so using Little's law $E(L) = \lambda E(S)$, we obtain that

$$E(L) = \frac{2}{3}(2 + p).$$