Midterm exam Stochastic Modelling

Vrije Universiteit Amsterdam
Faculty of Science

October 23, 2019, 12:00–14:00

This exam consists of six exercises, for which you can obtain 45 points in total (the distribution of points is indicated between square brackets). Your grade will be calculated as (number of points + 5)/5. Please write your name and student number on every paper you hand in, and give a clear and brief motivation of all your answers. The use of books or a graphical calculator during the exam is not allowed. Good luck!

1. [2 pt.] Give a precise definition of the *equilibrium distribution* of a continuous time Markov chain.

2. [5 pt.] Show that the n-step transition probabilities of a discrete time Markov chain satisfy the Chapman–Kolmogorov equations

$$p_{ij}^{(n+m)} = \sum_{k \in S} p_{ik}^{(n)} p_{kj}^{(m)} \qquad (n, m \geq 1).$$

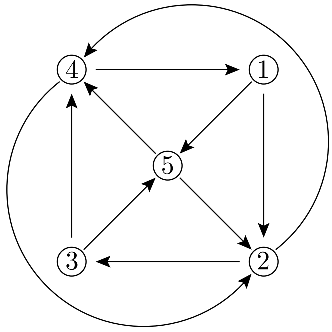
3. On the state space $S = \{1, 2, 3, 4, 5, 6\}$, consider the discrete time Markov chain with transition matrix

$$\mathrm{P} = \begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) [3 pt.] Draw the transition diagram of this Markov chain, determine the communicating classes of states, and for each class, indicate whether it is transient or absorbing.

(b) [5 pt.] Determine whether or not the Markov chain has a unique limiting distribution and/or a unique occupancy distribution, and if it does, determine the limiting and/or occupancy distribution.

4. Consider the small communication network depicted below:



When any of the five nodes in this network receives a message that is not targeted at that node, it forwards the message randomly (with equal probability) to one of the two nodes it is connected to (as indicated by the arrows). Suppose that node 2 forwards a message that is targeted at node 1.

(a) [5 pt.] Calculate the expected number of steps it takes (i.e., the expected number of times the message is forwarded) until the message reaches node 1.

(b) [5 pt.] Determine the probability that the message does not pass through node 5 before arriving at node 1.

5. A small supermarket has two checkouts. At each checkout, the service times follow an exponential distribution, but the service rate is $\frac{3}{2}\mu$ for the first checkout, and μ for the second. Customers arrive according to a Poisson process with a rate of λ customers per hour. Each customer chooses the first checkout with probability 2/3 (and hence the second with probability 1/3).

(a) [5 pt.] Determine the joint probability that at least two customers arrive at the first checkout between 9:00 and 10:30, and that no customers at all arrive between 9:30 and 10:00.

(b) [5 pt.] When you and your friend arrive, two customers are already being served (one at each checkout), and another customer is in the queue for the first checkout. You queue up for the first checkout, your friend for the second. Calculate the probability that you will finish checking out before your friend.

6. A specialised shop has two 3D printers that can be used by customers to print items. The time it takes to print a single item on one of the printers follows an exponential distribution, with an average printing time of half an hour. Customers arrive at the shop according to a Poisson process with an average of two customers per hour. Half of these customers require one printed item, the other half require two items. A customer who only needs one printed item utilizes any printer that is available when he arrives. A customers who wants to print two items occupies both machines if they are available; if only one printer is available, he prints his first item on that machine, and his second item on the first printer that becomes available. Customers who arrive while both machines are in use go elsewhere.

(a) [5 pt.] Formulate an appropriate continuous time Markov chain that can be used to model the utilization of the two 3D printers, and draw the transition rate diagram of this Markov chain.

(b) [5 pt.] Suppose that the operating costs per printer are 1 unit per hour when the printer is idle, and 5 units per hour when it is in use. Determine the shop's long-run operating costs per hour.