Solutions Midterm exam Stochastic Modeling

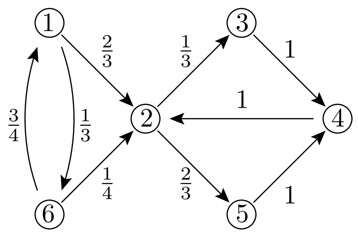
Vrije Universiteit Amsterdam
Faculty of Science

October 23, 2019

(Solutions to exercises 1 and 2 are in the course literature and/or lectures.)

3.

(a)



The communicating classes are

$\{1,6\}$	transient
$\{2,3,4,5\}$	absorbing

(b) The DTMC always ends up in the absorbing class $\{2,3,4,5\}$ eventually. On this class of states, the DTMC is positive recurrent (finite state space) and periodic with period 3. So there is no unique limiting distribution, but there is a unique occupancy distribution. The occupancy distribution $\hat{\pi}$ must satisfy the balance equations

$$\hat{\pi}_2 = \hat{\pi}_4 \qquad \hat{\pi}_3 = \tfrac{1}{3}\hat{\pi}_2$$
$$\hat{\pi}_4 = \hat{\pi}_3 + \hat{\pi}_5 \qquad \hat{\pi}_5 = \tfrac{2}{3}\hat{\pi}_2$$

and the normalizing equation $\hat{\pi}_2 + \hat{\pi}_3 + \hat{\pi}_4 + \hat{\pi}_5 = 1$. The solution is

$$\hat{\pi} = [\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4, \hat{\pi}_5, \hat{\pi}_6] = \left[0, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{3}, \tfrac{2}{9}, 0\right].$$

4.

(a) Let X_n be the position of the message in the network after it has been forwarded for the n-th time. Then $\{X_n, n = 0, 1, 2, \dots\}$ is a DTMC on the state space $\{1, 2, 3, 4, 5\}$. Define $m_i := E(T_1 \mid X_0 = i)$, where T_1 is the first-passage time to the state 1. We have to calculate m_2. By conditioning on the first step of the Markov chain, we conclude that the m_i must satisfy

$$m_2 = 1 + \tfrac{1}{2}m_3 + \tfrac{1}{2}m_4 \qquad m_3 = 1 + \tfrac{1}{2}m_4 + \tfrac{1}{2}m_5$$
$$m_4 = 1 + \tfrac{1}{2}m_2 \qquad m_5 = 1 + \tfrac{1}{2}m_2 + \tfrac{1}{2}m_4$$

Solving this sytem of equations for m_2 gives $m_2 = 6$.

(b) Make state 1 absorbing, and define $q_i := P(T_5 = \infty \mid X_0 = i)$, so that q_i is the probability that the DTMC, if it starts in state i, never visits state 5 before being absorbed in state 1. By conditioning on the first step of the Markov chain, we find that the q_i must satisfy

$$q_2 = \tfrac{1}{2}q_3 + \tfrac{1}{2}q_4 \qquad q_3 = \tfrac{1}{2}q_4 \qquad q_4 = \tfrac{1}{2} + \tfrac{1}{2}q_2$$

Solving this sytem of equations for q_2 gives $q_2 = \frac{3}{5}$.

5.

(a) The number of customers that arrive between 9:30 and 10:00 follows a Poisson distribution with parameter $\frac{1}{2}\lambda$. By thinning, customers arrive at the first checkout according to a Poisson process with rate $\frac{2}{3}\lambda$. The total number of customers arriving at the first checkout between 9:00 and 9:30 and between 10:00 and 10:30 is the sum of two independent Poisson random variables with parameter $\frac{1}{2} \times \frac{2}{3}\lambda$, and hence follows a Poisson distribution with parameter $\frac{2}{3}\lambda$.

Since the arrivals between 9:30 and 10:00 are independent from the arrivals between 9:00 and 9:30 and between 10:00 and 10:30, the desired joint probability is given by

$$e^{-\frac{1}{2}\lambda} \times \left(1 - (1 + \tfrac{2}{3}\lambda)e^{-\frac{2}{3}\lambda}\right).$$

(b) If all three customers at the first checkout complete service before the customer that is currently being served at the second checkout, then you are finished before your friend. The probability that this happens is

$$\left(\frac{\frac{3}{2}\mu}{\mu + \frac{3}{2}\mu}\right)^3 = \left(\frac{3}{5}\right)^3.$$

However, you also finish before your friend if first 0, 1 or 2 customers at the first checkout beat the customer that is currently being served at the second checkout, then the customer in service at the second checkout beats the next customer at the first checkout, and finally, the remaining customer(s) at the first checkout complete service before your friend (who is then in service at the second checkout). This has probability

$$\sum_{k=0}^{2} \left(\frac{\frac{3}{2}\mu}{\mu + \frac{3}{2}\mu} \right)^{k} \times \frac{\mu}{\mu + \frac{3}{2}\mu} \times \left(\frac{\frac{3}{2}\mu}{\mu + \frac{3}{2}\mu} \right)^{3-k} = 3 \times \left(\frac{3}{5}\right)^{3} \times \frac{2}{5},$$

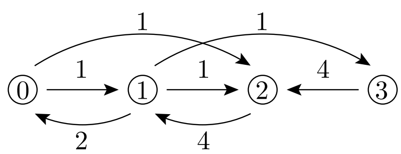
where the sum is over the number k of customers at the first checkout that beat the customer who is initially in service at the second checkout.

Combining the probabilities computed above, we conclude that

$$P(\text{you finish before friend}) = \left(\frac{3}{5}\right)^{3} + 3 \times \left(\frac{3}{5}\right)^{3} \times \frac{2}{5} = \frac{11}{5}\left(\frac{3}{5}\right)^{3}.$$

6.

(a) Let $X(t)$ denote the number of items in the shop that are either being printed or waiting to be printed. Then $\{X(t), t \geq 0\}$ is a CTMC on the state space $\{0, 1, 2, 3\}$ with transition rate diagram



Note that with this choice of state space, both printers are occupied if the CTMC is in state 2 or 3, one printer is in use when the CTMC is in state 1, and both printers are idle if the state is 0.

(b) The balance equations for this CTMC are

$$2p_0 = 2p_1 \qquad 4p_1 = p_0 + 4p_2 \qquad 4p_2 = p_0 + p_1 + 4p_3 \qquad 4p_3 = p_1$$

Solving the balance equations and normalizing equation gives

$$\mathrm{p} = [p_0, p_1, p_2, p_3] = \left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{12}\right].$$

Hence, by the ergodic theorem, the long-run operating costs per hour are

$$2 \times p_0 + 6 \times p_1 + 10 \times (p_2 + p_3) = (2 + 6 + 10) \times \tfrac{1}{3} = 6.$$