Resit exam Stochastic Modeling (X_400646)

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Faculty of Science

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This exam consists of four exercises, for which you can obtain a total of 45 points. The number of points you can obtain for each part of an exercise is indicated between square brackets. The grade for the exam is determined by the formula (number of points + 5)/5.

Please write your name and student number on every paper you hand in, and ***motivate all your answers***, even if the question does not explicitly ask for this. The use of books or a graphical calculator is not allowed. A formula sheet is included on the last page of the exam.

Good luck!

1. Consider a discrete time Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) [3 pt.] Draw the transition diagram of this Markov chain, determine the classes of communicating states, and decide for each of these classes whether it is transient or absorbing.

(b) [3 pt.] Determine whether or not the limiting distribution exists and whether or not the occupancy distribution exists, and if it exists, determine the limiting and/or occupancy distribution.

(c) [4 pt.] Denote by q_i $(i = 1, 2, \ldots, 6)$ the probability that the Markov chain will (at any time) visit state 6, if the Markov chain starts in state i (where $q_6 = 1$). Determine q_1, i.e. the probability that the Markov chain will visit state 6 when it starts in state 1.

2. An investment company considers an investment to be *good* if it returns a profit, and *bad* if it leads to a loss. Suppose investments are good with probability p and bad with probability $1-p$, independently of each other. The company wants to avoid following up a good investment with a string of bad investments. To investigate the probability that such an event happens, they decide to model their investment history using a discrete time Markov chain using the state space $\{0, 1, 2, 3, \dots\}$, where state 0 means that the last investment was good, and state $i > 0$ means that the last i investments were bad and were preceded by a good investment. We assume that the company starts with a good investment (i.e. in state 0).

(a) [2 pt.] Draw the transition diagram for the Markov chain describing the history of investments, using the state space described above.

(b) [4 pt.] For $i > 0$, express the equilibrium probability π_i that the Markov chain is in state i in terms of π_0, and use this to explicitly determine the equilibrium distribution.

(c) [4 pt.] Take $p = 4/5$. Determine the expected number of investments after the initial good investment until the company has (for the first time) made a good investment followed by three consecutive bad investments.

3. Consider an M/M/2/4 queue with an arrival rate μ that is equal to the service rate λ, i.e. $\mu = \lambda$. Customers who arrive when all places in the system are occupied, are blocked.

(a) [4 pt.] Let p_i be the equilibrium probability that there are i customers in the system. Draw the state diagram with transition rates, and show that for $i = 1, 2, \dots, 4$, $p_i = 2^{1-i} p_0$.

(b) [2 pt.] Determine the probability that an arriving customer is blocked.

(c) [4 pt.] Calculate the expected number of customers in the system, and the expected sojourn time of customers who are not blocked.

4. A server processes two kinds of data packets: small and large. Small packets have a fixed size, and therefore a fixed service time of a milliseconds. Large packets, on the other hand, have a variable size and a service time with a mean of $5a$ milliseconds, and a squared coefficient of variation equal to $11/25$. Small packets arrive according to a Poisson process with rate $2/(5a)$, and large packets arrive according to a Poisson process with rate $2/(25a)$. Packets are processed on a First Come First Served basis.

(a) [2 pt.] What is the probability that exactly three small packets will arrive within the next 20 milliseconds?

(b) [3 pt.] Calculate the probability that the next two data packets that arrive are of different kinds and arrive within 25 milliseconds from each other.

(c) [5 pt.] Calculate the expected time an arbitrary arriving packet has to wait before receiving service.

To handle packets more efficiently, the company deploying the server considers changing the service discipline in the following way:

1. small packets are served FCFS, and take priority over large packets (i.e. they join the queue immediately in front of any large packets);
2. large packets are also served FCFS, but their service does not start until there are no more small packets in the system;
3. the service of any packet (small or large) is never interrupted.

(d) [5 pt.] Argue that the expected waiting time of small packets $E(W^q_{\text{small}})$ and expected number of small packets in the queue $E(L^q_{\text{small}})$ are related via the arrival relation

$$E(W^q_{\text{small}}) = aE(L^q_{\text{small}}) + \rho E(R),$$

where ρ is the overall load of the system, and $E(R)$ the residual service time of an arbitrary packet in service if the system is non-empty. Use this relation to determine the expected waiting time of a small packet. Give an equivalent arrival relation for the case that small packets are served LCFS, and use this relation to determine the expected waiting time of a small packet in the LCFS case.

FORMULA SHEET

Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$

$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$

$$P(S > t) = \frac{\Pi_W}{1 - c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1 - c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khinchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$

$$E(BP) = \frac{E(B)}{1-\rho}.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0, 1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$