Solutions Resit exam Stochastic Modeling (X_400646)

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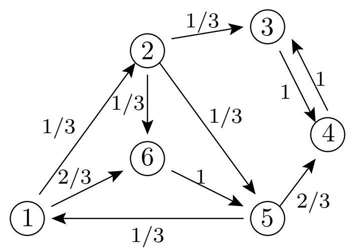
1.

(a) [3 pt.] The transition diagram looks as follows:



The classes of communicating states are

$$\begin{array}{ll} \{1,2,5,6\} & \text{transient} \\ \{3,4\} & \text{absorbing} \end{array}$$

(b) [3 pt.] The states in the absorbing class $\{3,4\}$ are periodic with period 2, hence the limiting distribution does not exist.The occupancy distribution does exist, because the DTMC is positive recurrent on the absorbing class (we have a finite state space). The equilibrium equations for the occupancy distribution are simply

$$\hat{\pi}_3 = \hat{\pi}_4,$$

so the occupancy distribution is

$$\hat{\pi} = [\hat{\pi}_1, \ldots, \hat{\pi}_6] = [0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0].$$

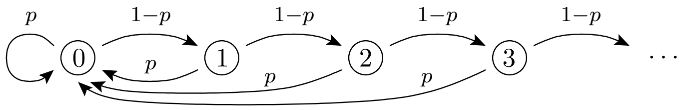
(c) [4 pt.] By conditioning on the first step of the DTMC, and noting that $q_3 = q_4 = 0$, we see that the q_i must satisfy the following set of equations:

$$\begin{aligned} q_1 &= \tfrac{1}{3}q_2 + \tfrac{2}{3} \\ q_2 &= \tfrac{1}{3}q_5 + \tfrac{1}{3} \\ q_5 &= \tfrac{1}{3}q_1 \end{aligned}$$

Solving these equations for q_1 gives $q_1 = \frac{21}{26}$.

2.

(a) [2 pt.] The transition diagram looks as follows:



(b) [4 pt.] The equilibrium equations for the states $i > 0$ read

$$\pi_i = (1 - p)\pi_{i-1},$$

from which it follows that $\pi_i = (1 - p)^i \pi_0$ for all $i > 0$.Normalization gives

$$1 = \sum_{i=0}^{\infty} (1 - p)^i \pi_0 = \frac{1}{p}\,\pi_0,$$

hence $\pi_0 = p$ and $\pi_i = (1 - p)^i p$ for $i \geq 1$.

(c) [4 pt.] Let m_i denote the expected number of steps until the DTMC reaches state 3, when the DTMC starts in state i. Our task is to find m_0. By conditioning on the first step of the DTMC, we see that the m_i satisfy the following set of equations:

$$\begin{aligned} m_0 &= 1 + \tfrac{4}{5} m_0 + \tfrac{1}{5} m_1 \\ m_1 &= 1 + \tfrac{4}{5} m_0 + \tfrac{1}{5} m_2 \\ m_2 &= 1 + \tfrac{4}{5} m_0 \end{aligned}$$

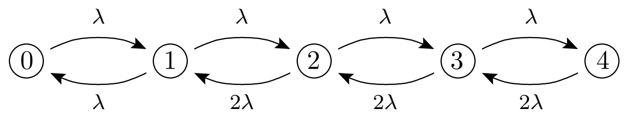
Possibly the easiest way to solve these equations is to multiply the third by 5, the second by 25, and the first by 125, which gives

$$\begin{aligned} 125 m_0 &= 125 + 100 m_0 + 25 m_1 \\ 25 m_1 &= 25 + 20 m_0 + 5 m_2 \\ 5 m_2 &= 5 + 4 m_0 \end{aligned}$$

Substituting the third equation into the second, and then the second into the first, yields $m_0 = 155$.

3.

(a) [4 pt.] The state diagram looks as follows:



Using the 'global balance' principle, the balance equations are

$$\lambda p_0 = \lambda p_1$$
$$\lambda p_{i-1} = 2\lambda p_i \quad i = 2, 3, 4$$

It follows that $p_1 = p_0$ and $p_i = \frac{1}{2} p_{i-1}$ for $i = 2, 3, 4$, so that

$$p_i = 2^{1-i}\, p_0 \qquad \text{for } i = 1, 2, 3, 4.$$

(b) [2 pt.] Normalization: $1 = \sum_{i=0}^{4} p_i = p_0 + (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) p_0 = \frac{23}{8}\, p_0$ gives $p_0 = 8/23$, from which it follows that the probability that an arriving customer is blocked, by PASTA, is $p_4 = \frac{1}{8} p_0 = 1/23$.

(c) [4 pt.] We have $E(L) = \sum_{i=1}^{4} i p_i$. A short calculation gives $E(L) = \frac{26}{23}$. We now have (at least) two different approaches at our disposal for finding the expected sojourn time $E(S)$ of customers who are not blocked:

1. The arrival rate of customers who are not blocked is $\frac{22}{23}\lambda$, so Little's Law tells us that $E(L) = \frac{22}{23}\lambda E(S)$, from which it follows that

$$E(S) = \frac{23}{22}\,\frac{1}{\lambda}\, E(L) = \frac{13}{11}\,\frac{1}{\lambda}.$$

2. Alternatively, an arriving customer has to wait to receive service when there are either two or three customers in the system upon arrival. The conditional probabilities of finding two or three customers upon arrival, given that the arriving customer is not blocked, are $\frac{23}{22} p_2 = \frac{2}{11}$ and $\frac{23}{22} p_3 = \frac{1}{11}$, respectively, while the expected waiting time in these two cases is $1/(2\lambda)$ or $2/(2\lambda)$, respectively. Hence the expected sojourn time of a customer who is not blocked, is

$$E(S) = \left[\frac{2}{11} \cdot \frac{1}{2\lambda} + \frac{1}{11} \cdot \frac{2}{2\lambda}\right] + \frac{1}{\lambda} = \frac{13}{11}\,\frac{1}{\lambda}.$$

4.

(a) [2 pt.] The number of small packets that will arrive in the next 20 milliseconds has a Poisson distribution with parameter $\frac{2}{5a} \cdot 20 = 8/a$. Hence, the probability that exactly three small packets will arrive within the next 20 milliseconds is

$$\frac{(8/a)^3}{3!} e^{-8/a}.$$

(b) [3 pt.] By superposition of Poisson processes, packets arrive according to a Poisson process with a total rate of $2/(5a) + 2/(25a) = 12/(25a)$. With probability $1/6$, an arriving packet is large, and with probability $5/6$, an arriving packet is small. The probability that the next two arriving packets are of different kinds is therefore $\frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{18}$.

The interarrival time between these two packets is independent of the time until the first of them arrives, and has an exponential distribution with parameter $12/(25a)$. Therefore, the probability that the next two data packets that arrive are of different kinds *and* arrive within 25 milliseconds from each other is given by

$$\frac{5}{18} \cdot \left(1 - e^{-25 \cdot 12/(25a)}\right) = \frac{5}{18} \cdot \left(1 - e^{-12/a}\right).$$

(c) [5 pt.] We are dealing with an M/G/1 system, so the expected waiting time is given by the Pollaczek–Khinchine formula

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)}.$$

Since arriving packets are small with probability $5/6$ and large with probability $1/6$, we have that

$$E(B) = \frac{5}{6} \cdot a + \frac{1}{6} \cdot 5a = \frac{5}{3}a,$$

so that $\rho = \lambda E(B) = \frac{12}{25a} \cdot \frac{5}{3}a = \frac{4}{5}$.

Next we need to compute $E(B^2)$. First we observe that for large packets,

$$E(B_{\text{large}}^2) = (c_{B_{\text{large}}}^2 + 1)(EB_{\text{large}})^2 = \frac{36}{25} \cdot 25a^2 = 36a^2.$$

Therefore,

$$E(B^2) = \frac{5}{6} \cdot a^2 + \frac{1}{6} \cdot 36a^2 = \frac{41}{6}a^2.$$

Substituting this into the Pollaczek–Khinchine formula yields

$$E(W^q) = \frac{4/5}{1/5} \cdot \frac{41a^2}{2 \cdot 6 \cdot (5/3)a} = \frac{41}{5}a.$$

(d) [5 pt.] A small packet that arrives has to wait for 1) the service of any small packets that are already waiting in the queue, 2) the residual service time of any packet (large or small) that is in service upon arrival if the server is busy. This explains the two terms in the relation

$$E(W^q_{\text{small}}) = a\,E(L^q_{\text{small}}) + \rho\,E(R).$$

Little's Law for small packets says that $E(L^q_{\text{small}}) = \frac{2}{5a}E(W^q_{\text{small}})$. Substituting this into the arrival relation, and using our results from part (c), we find that

$$E(W^q_{\text{small}}) = \frac{5}{3}\rho\,E(R) = \frac{5}{3}\cdot\frac{4}{5}\cdot\frac{E(B^2)}{2E(B)} = \frac{4}{3}\cdot\frac{41a^2}{2\cdot 6\cdot(5/3)a} = \frac{41}{15}a.$$

When small packets are served LCFS, an arriving small packet has to wait for the residual service time of any packet that is in service upon arrival, but also for the entire "busy period" of any small packets that arrive during this residual service time (where the busy periods consists of handling small packets only). Therefore, the arrival relation in the LCFS case is

$$E(W^q_{\text{small}}) = \rho\,E(R) + \frac{2}{5a}\rho\,E(R)\cdot\frac{a}{1-\rho_{\text{small}}},$$

where ρ_{small} is the fraction of time the server is working on small packets, that is, $\rho_{\text{small}} = \frac{2}{5a}\cdot a = \frac{2}{5}$. Substituting this value and simplifying gives

$$E(W^q_{\text{small}}) = \frac{5}{3}\rho\,E(R),$$

which shows that the expected waiting time of small packets is the same in the LCFS case as in the FCFS case.