Final exam Stochastic Modeling

Vrije Universiteit Amsterdam
Faculty of Science

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This exam consists of three exercises, divided into ten parts, for which you can obtain a total of 36 points. The number of points you can obtain for each part of an exercise is indicated between square brackets. The grade for this exam is determined by the formula (number of points + 4)/4.

Please write your name and student number on every paper you hand in, and give a clear and brief motivation of all your answers. The use of books or a graphical calculator is not allowed.

Good luck!

1. At a new container terminal that has recently been opened in Rotterdam harbour, containers arrive according to a Poisson process with an average of 15 containers per day. The average time a container stays in the terminal is 3 days. The terminal currently has more than enough space for containers, so we can treat it for now as having infinite storage capacity.

(a) [3 pt.] Assume that the time a container stays in the terminal follows an exponential distribution. Formulate an appropriate continuous time Markov chain for the number of containers stored in the terminal, and draw the state diagram with transition rates.

(b) [4 pt.] Formulate the balance equations corresponding to the diagram from part (a), and explicitly derive the equilibrium distribution of the number of containers in the terminal from these balance equations.

The container terminal actually has room for 100 containers, and in reality, the time a container stays in the terminal has an unknown distribution with a mean of 3 days. The arrival rate of containers is expected to grow to an average of 30 containers per day. If a container arrives when the terminal is full, the container is blocked and stored elsewhere.

(c) [3 pt.] What is the equilibrium distribution for the number of containers in the terminal in this case, and what is the probability that an arriving container will be blocked? Please explain your answers.

2. Consider a queue with a single server, where customers arrive according to a Poisson process with rate $3/(2b)$, and service times are uniformly distributed on the interval $(0, b)$, with $b > 0$.

(a) [3 pt.] Suppose that a customer arrives at an arbitrary moment when the server is busy. What is the probability that it will take at most time t until the current service has been completed?

(b) [4 pt.] Show that the expected time that an arriving customer has to wait is $E(W^q) = b$.

Suppose now that there are two types of customers: customers of type 1 and 2 arrive according to independent Poisson processes with respective rates $\lambda_1 = 3/4$ and $\lambda_2 = 3/(2b)$, with $b > 0$. We want to compare the following two situations:

A. **(No pooling)** We have a separate queue and server for each type of customer as in parts (a)–(b) above. The service times are uniform on $(0, 2)$ for type 1, and uniform on $(0, b)$ for type 2.

B. **(Pooling)** Customers share one queue and one server with twice the service capacity of the servers in situation A, so that the service times are now uniform on $(0, 1)$ for type 1 and uniform on $(0, \frac{1}{2}b)$ for type 2.

(c) [5 pt.] Calculate the expected waiting time of customers in situation B, and determine the range of values of b for which *both* types of customer have a smaller expected waiting time in situation B than in situation A.

3. In a queueing system, customers arrive according to a Poisson process with rate λ, and are served by a single server. The service times have an exponential distribution with rate μ. Furthermore, if there are no customers in the system, the server works on other tasks. If a new customer arrives while the server is busy with other tasks, it takes the server an exponentially distributed amount of time with parameter θ to switch back to serving customers.

(a) [4 pt.] Formulate the arrival relation between the expected waiting time $E(W^q)$ and the expected number of customers in the queue $E(L^q)$. Use this relation to determine $E(L^q)$ explicitly in terms of the parameters λ, μ and θ.

The state of the system can be described by the random variables

$$L(t) = \#\text{ of customers in system at time } t,$$
$$Y(t) = \#\text{ of customers in service at time } t,$$

where $L(t) \in \{0, 1, 2, \dots\}$, and $Y(t)$ is 1 if the server is serving customers, and 0 if the server is working on other tasks.

(b) [4 pt.] Draw the state diagram with transition rates for the continuous time Markov chain $\{(L(t), Y(t)), t \geq 0\}$, and formulate the detailed balance equations corresponding to this diagram.

(c) [2 pt.] For what values of the parameters λ, μ and θ is the system stable?

Let $p_{i,0}$ and $p_{i,1}$ denote the respective equilibrium probabilities of finding the system in state $(i, 0)$ and $(i, 1)$. Suppose that the system is stable and $\mu \neq \lambda + \theta$. We claim that then the equilibrium distribution is given by

$$p_{i,0} = \frac{\mu - \lambda - \theta}{\lambda + \theta} \left(\frac{\lambda}{\lambda + \theta} \right)^i C \qquad (i \geq 0),$$

$$p_{i,1} = \left[\left(\frac{\lambda}{\lambda + \theta} \right)^i - \left(\frac{\lambda}{\mu} \right)^i \right] C \qquad (i \geq 1),$$

with C a normalizing constant (NB: you do not have to prove this).

(d) [4 pt.] Assume that the equilibrium distribution indeed has the form given above. Calculate the constant C, and show that the probability that an arriving customer finds the system in the empty state $(0, 0)$ is

$$p_{0,0} = \frac{\theta}{\mu} \frac{\mu - \lambda}{\lambda + \theta}.$$

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Erlang distribution. If S_n has an Erlang(n, μ) distribution, then

$$P(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!} \qquad \text{and} \qquad f_{S_n}(t) = \mu\, e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!}.$$

M/M/c queue. The probability of waiting Π_W, waiting time W^q and sojourn time S satisfy

$$\Pi_W = \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!},$$
$$E(W^q) = \Pi_W \frac{1}{c\mu(1-\rho)} \qquad \text{and} \qquad P(W^q > t) = \Pi_W\, e^{-c\mu(1-\rho)t},$$
$$P(S > t) = \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}.$$

M/G/c/c queue. The blocking probability is

$$B(c, a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!} \qquad \text{with } a = \lambda\, E(B) = c\rho.$$

M/G/1 queue. The waiting time W^q for FCFS (Pollaczek–Khintchine) and the busy period BP satisfy

$$E(W^q) = \frac{\rho}{1-\rho} \frac{E(B^2)}{2E(B)} = \frac{1}{2} \frac{\rho}{1-\rho}(1 + c_B^2)E(B),$$
$$E(BP) = \frac{E(B)}{1-\rho}.$$

G/M/1 queue. The limiting probability a_i of finding i customers upon arrival, and the expected waiting time $E(W^q)$ are given by

$$a_i = (1-\sigma)\sigma^i \qquad \text{and} \qquad E(W^q) = \frac{\sigma}{\mu(1-\sigma)},$$

where σ is the unique solution in $(0, 1)$ of $\sigma = E\left[e^{-\mu(1-\sigma)A}\right]$ with A an interarrival time.

Residual life time. Let X be a life time and R the corresponding residual life time. Then

$$P(R \le x) = \frac{1}{E(X)} \int_0^x P(X > u)\, \mathrm{d}u \qquad \text{and} \qquad E(R) = \frac{E(X^2)}{2E(X)}.$$