Solutions Final exam Stochastic Modeling

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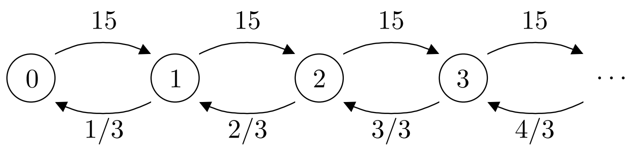
1.

(a) Let $X(t) = \#$ of containers in the terminal at time t. Then the CTMC $\{X(t), t \geq 0\}$ behaves as an M/M/∞ queue, with arrival rate $\lambda = 15$ per day, and service rate $\mu = 1/3$ per day. The state space is $I = \{0, 1, 2, \ldots\}$.

The transition rate diagram is as follows:



(b) Using the "global balance principle", the balance equations are

$$15p_{i-1} = (i/3)p_i, \qquad i = 1, 2, 3, \ldots$$

Therefore, for $i \geq 1$,

$$p_i = \frac{15}{i/3}\, p_{i-1} = \frac{45}{i}\, p_{i-1} = \frac{45}{i}\,\frac{45}{i-1}\, p_{i-2} = \cdots = \frac{45^i}{i!}\, p_0.$$

This is also valid for $i = 0$. The normalizing equation therefore tells us that

$$\sum_{i=0}^{\infty} p_i = \sum_{i=0}^{\infty} \frac{45^i}{i!}\, p_0 = e^{45}\, p_0 = 1.$$

Hence

$$p_0 = e^{-45} \qquad \text{and} \qquad p_i = e^{-45}\,\frac{45^i}{i!}.$$

In other words, the equilibrium distribution of the number of containers in the terminal is Poisson with parameter 45.

(c) The situation is now described by an M/G/c/c queue (an Erlang-B model), with capacity $c = 100$, arrival rate $\lambda = 30$, and mean service time $E(B) = 3$. The total load is then $a = \lambda\, E(B) = 90$.

The equilibrium distribution of the Erlang-B model is insensitive to the service time distribution. Therefore, we can repeat (a)–(b) to determine the equilibrium distribution, with $\lambda = 30$ instead of 15, and state space restricted to $I = \{0, 1, 2, \ldots, 100\}$. Hence the equilibrium distribution is

$$p_i = \frac{90^i}{i!}\, p_0 \qquad (i \in I), \qquad \text{with} \qquad p_0 = \left[\sum_{i=0}^{100} \frac{90^i}{i!}\right]^{-1}.$$

The blocking probability is given by

$$p_{100} = B(90, 100) = \frac{90^{100}/100!}{\sum_{i=0}^{100} 90^i/i!}.$$

2.

(a) The time until the current service has been completed is the residual service time R corresponding to the service time B. It satisfies

$$P(R \le t) = \frac{1}{E(B)} \int_0^t P(B > u)\,\mathrm{d}u.$$

Since B is uniform on $(0, b)$, we have $E(B) = b/2$ and $P(B > u) = (b - u)/b$ for $u \in (0, b)$. Therefore, for $t \in (0, b)$,

$$P(R \le t) = \frac{2}{b} \int_0^t \frac{b-u}{b}\,\mathrm{d}u = \frac{2}{b^2}\left[bu - \tfrac{1}{2}u^2\right]_0^t = \frac{2}{b^2}\left(bt - \tfrac{1}{2}t^2\right).$$

Finally, for $t \ge b$, $P(R \le t) = 1$, of course.

(b) The system is an M/G/1 queue, for which the expected waiting time is given by the Pollaczek–Khintchine formula

$$E(W^q) = \frac{\rho}{1-\rho}\frac{E(B^2)}{2E(B)}.$$

We have $\lambda = 3/(2b)$ and $E(B) = b/2$, so that the load is $\rho = \lambda\, E(B) = 3/4$. Furthermore, the second moment of the service time is

$$E(B^2) = \int_0^b \frac{x^2}{b}\,\mathrm{d}x = \frac{1}{3}\frac{x^3}{b}\Big|_0^b = \frac{1}{3}b^2.$$

Substituting all this into the formula for the expected waiting time, we obtain

$$E(W^q) = \frac{3/4}{1/4}\,\frac{b^2/3}{b} = b.$$

(c) In the pooling situation, customers arrive according to a Poisson process with rate $\lambda_1+\lambda_2 = \frac{3}{4}+\frac{3}{2b} = \frac{3(b+2)}{4b}$. The probability that an arriving customer is of type 1 is

$$p_1 = \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{3}{4}\frac{4b}{3(b+2)} = \frac{b}{b+2},$$

so the probability that an arriving customer is of type 2 is $p_2 = 1-p_1 = \frac{2}{b+2}$.

Since the service times are uniform on $(0,1)$ for type 1 and uniform on $(0,\frac{1}{2}b)$ for type 2, the expected service time of a customer is

$$E(B) = p_1 \times \frac{1}{2} + p_2 \times \frac{b}{4} = \frac{b}{b+2}\frac{1}{2} + \frac{2}{b+2}\frac{b}{4} = \frac{b}{b+2}.$$

Therefore, the load is $\rho = \lambda\, E(B) = 3/4$ (as before).

From part (b) we see that the second moment of a random variable that is uniform on $(0,d)$ is $d^2/3$. Hence

$$E(B^2) = p_1 \times \frac{1}{3} + p_2 \times \frac{b^2}{12} = \frac{b}{b+2}\frac{1}{3} + \frac{2}{b+2}\frac{b^2}{12} = \frac{2b+b^2}{6(b+2)} = \frac{b(b+2)}{6(b+2)} = \frac{b}{6}.$$

Combining everything in the Pollaczek–Khintchine formula, we get

$$E(W^q) = \frac{3/4}{1/4}\frac{b/6}{2b/(b+2)} = \frac{b+2}{4}.$$

From part (b) we know that in the "no pooling" situation, the expected waiting times of customers of type 1 and 2 are, respectively, 2 and b. So both types of customers are better off in the "pooling" situation if and only if

$$\frac{b+2}{4} < 2 \qquad \text{and} \qquad \frac{b+2}{4} < b,$$

or in other words, when $b < 6$ and $b > 2/3$.

3.

(a) If the arrival process is Poisson with rate λ, service times have an exponential distribution with parameter μ, and the system is stable, then the fraction of time the server is busy serving customers must be $\rho = \lambda/\mu$. An arriving customer has to wait for the service of all customers who are waiting in the queue. Moreover, if the server is serving a customer upon arrival, he also has to wait for completion of this service, and otherwise he has to wait for the server to switch from his current task to serving customers. Hence, the arrival relation is

$$E(W^q) = E(L^q)\frac{1}{\mu} + \rho\frac{1}{\mu} + (1-\rho)\frac{1}{\theta}.$$

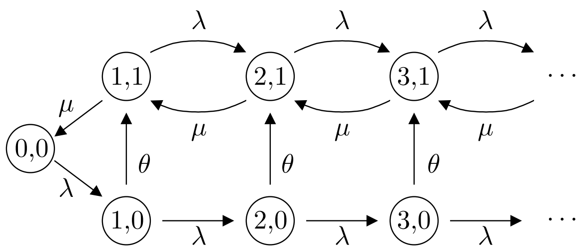
Using Little's law $E(L^q) = \lambda E(W^q)$, we now get

$$E(L^q) = \lambda E(W^q) = \rho E(L^q) + \rho^2 + (1-\rho)\frac{\lambda}{\theta}.$$

Solving for $E(L^q)$ finally gives

$$E(L^q) = \frac{\rho^2}{1-\rho} + \frac{\lambda}{\theta}.$$

(b) The transition rate diagram looks as follows:



The (detailed) balance equations are

$$\begin{aligned}
\lambda p_{0,0} &= \mu p_{1,1} \\
(\lambda+\mu) p_{1,1} &= \theta p_{1,0} + \mu p_{2,1} \\
(\lambda+\theta) p_{i,0} &= \lambda p_{i-1,0} && (i = 1,2,3,\dots) \\
(\lambda+\mu) p_{i,1} &= \theta p_{i,0} + \lambda p_{i-1,1} + \mu p_{i+1,1} && (i = 2,3,4,\dots)
\end{aligned}$$

(c) Observe that as long as $\theta > 0$, the Markov chain can never get "stuck" on the states $(i,0)$, $i \geq 0$: it will always (eventually) reach one of the states where $Y(t) = 1$. Once the Markov chain is in this part of the state space, it moves "up" at rate λ and "down" at rate μ until it reaches $(0,0)$ again. To make the system stable, it therefore suffices that $\theta > 0$ and $\mu > \lambda$ (since the latter prevents the number of customers in the system from blowing up).

(d) The constant C is determined by the normalizing equation:

$$\sum_{i=0}^{\infty} p_{i,0} + \sum_{i=1}^{\infty} p_{i,1} = 1.$$

Noting that the formula for $p_{i,1}$ reduces to zero for $i = 0$, we can actually let the second sum start from $i = 0$ as well, which gives us

$$
\begin{aligned}
1 = \sum_{i=0}^{\infty} \big(p_{i,0} + p_{i,1}\big) &= \sum_{i=0}^{\infty} \left[\frac{\mu - \lambda - \theta}{\lambda + \theta} \left(\frac{\lambda}{\lambda + \theta} \right)^i + \left(\frac{\lambda}{\lambda + \theta} \right)^i - \left(\frac{\lambda}{\mu} \right)^i \right] C \\
&= \sum_{i=0}^{\infty} \left[\frac{\mu}{\lambda + \theta} \left(\frac{\lambda}{\lambda + \theta} \right)^i - \left(\frac{\lambda}{\mu} \right)^i \right] C \\
&= \left[\frac{\mu}{\lambda + \theta} \frac{1}{1 - \lambda/(\lambda + \theta)} - \frac{1}{1 - \lambda/\mu} \right] C \\
&= \left[\frac{\mu}{\lambda + \theta} \frac{\lambda + \theta}{\theta} - \frac{\mu}{\mu - \lambda} \right] C \\
&= \frac{\mu(\mu - \lambda) - \mu\theta}{\theta(\mu - \lambda)} C \\
&= \frac{\mu}{\theta} \frac{\mu - \lambda - \theta}{\mu - \lambda} C.
\end{aligned}
$$

Therefore,

$$
C = \frac{\theta}{\mu} \frac{\mu - \lambda}{\mu - \lambda - \theta} \qquad \text{hence} \qquad p_{0,0} = \frac{\mu - \lambda - \theta}{\lambda + \theta} C = \frac{\theta}{\mu} \frac{\mu - \lambda}{\lambda + \theta}.
$$