Midterm exam Stochastic Modeling

Vrije Universiteit Amsterdam
Faculty of Science

October 24, 2018, 12:00–14:00

This midterm exam consists of four exercises, for which you can obtain a total of 36 points. The number of points you can obtain for each part of an exercise is indicated between square brackets. The midterm grade is determined by the formula (number of points + 4)/4.

Please write your name and student number on every paper you hand in, and give a clear and brief motivation of all your answers. The use of books or a graphical calculator is not allowed.

Good luck!

1. Consider a discrete time Markov chain on the state space $\{1, 2, 3, 4, 5\}$ with transition matrix

$$\mathrm{P} = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) [2 pt.] Draw the transition diagram of this Markov chain, and determine the classes of communicating states.

(b) [4 pt.] Determine whether or not the limiting distribution exists and whether or not the occupancy distribution exists, and if it exists, determine the limiting and/or occupancy distribution.

(c) [3 pt.] Determine the expected number of steps it takes the Markov chain to reach state 3, if the Markov chain starts in state 1.

2. A bike rental shop in Amsterdam also repairs bikes that are returned broken. The shop is able to repair two bikes in a day, so for efficiency reasons, the shop waits until two broken bikes have been returned, before starting with the repairs the next day. On any given day, the probability that a broken bike is returned is p, and the probability that more than one broken bike is returned is so small, that it can be neglected.

(a) [3 pt.] Formulate an appropriate discrete time Markov chain to model the development of the number of broken bikes waiting to be repaired at the end of each day. Draw the transition diagram of the Markov chain, and specify the one-step transition probabilities.

(b) [4 pt.] Determine the long-run fraction of days on which repairs are carried out, by specifying and solving the equilibrium equations.

On closer examination, it turns out that with probability 0.1, the shop does not manage to repair two bikes in a single day, but instead only manages to repair one bike.

(c) [3 pt.] Modify the Markov chain to model the new situation, and draw the new transition diagram.

3. The task scheduler in a computer system maintains two separate queues, one for jobs with a low priority, and one for high-priority jobs. Jobs arrive at the task scheduler according to a Poisson process with an average of λ jobs per millisecond. Of these jobs, 20% have a high priority.

(a) [3 pt.] What is the probability that in the next 5 milliseconds, *exactly* one high-priority job and *at most* two low-priority jobs arrive at the task scheduler?

The task scheduler is responsible for sending jobs to the central processor for execution. The central processor can execute one job at a time. The time it takes to complete a job follows an exponential distribution with rate μ.

(b) [4 pt.] Suppose that the processor is currently processing a job, and that exactly two low-priority jobs and no high-priority jobs are waiting in line at the task scheduler. What is the probability that the next job that arrives at the task scheduler is a high-priority job, and that it arrives while the processor is executing the last low-priority job that was waiting in the queue?

4. A factory has three production lines, each of which can be either *Up* (functioning) or *Down* (in repair). When all three production lines are Up, the factory uses only two of them and keeps the third production line on stand-by to take over production when one of the other lines goes Down. In all other situations, the factory uses all the production lines that are Up. If a production line is Up and in use, it goes Down after a time that follows an exponential distribution with rate λ. A production line that is Down, goes Up again after a time that is exponential with rate μ.

(a) [3 pt.] Formulate an appropriate continuous time Markov chain that can be used to model the number of production lines that are Up, and draw the transition rate diagram of this Markov chain.

(b) [3 pt.] Suppose that initially (i.e. at time 0) all three production lines are Up, and the factory has two production lines in use. Determine the expected time it takes until the factory has only one production line Up and running (in terms of the parameters λ and μ).

Suppose that for each production line that is in use, the production costs amount to c per unit of time. Furthermore, the repair costs are k per unit of time for each production line that is Down.

(c) [4 pt.] Determine the long-run average costs per time unit, expressed explicitly in terms of the parameters λ and μ.