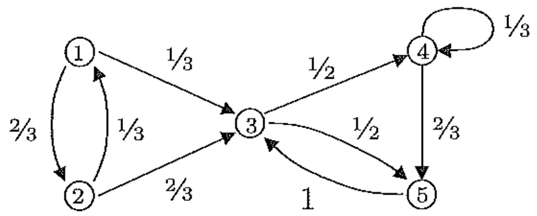
Solutions Midterm exam Stochastic Modeling

Vrije Universiteit Amsterdam
Faculty of Science

October 24

1.

(a) [2 pt.]



1 pt

The communicating classes are

$$\begin{array}{ll} \{1,2\} & \text{transient} \\ \{3,4,5\} & \text{absorbing} \end{array}$$

1 pt

(b) [4 pt.] The DTMC will always end up in the absorbing class $\{3,4,5\}$, and on this class, the DTMC is aperiodic and positive recurrent (finite state space). Hence, the limiting and occupancy distributions both exist (and are equal). The equilibrium equations are 1 pt

$$\pi_3 = \pi_5 \qquad \pi_4 = \tfrac{1}{3}\pi_4 + \tfrac{1}{2}\pi_3 \qquad \pi_5 = \tfrac{1}{2}\pi_3 + \tfrac{2}{3}\pi_4$$

1 pt

The second equation gives $\pi_4 = \frac{3}{4}\pi_3$. Normalization says that

$$\pi_3 + \pi_4 + \pi_5 = \pi_3\big(1 + \tfrac{3}{4} + 1\big) = \tfrac{11}{4}\pi_3 = 1,$$

hence $\pi_3 = {}^4/_{11}$, $\pi_4 = {}^3/_{11}$, $\pi_5 = {}^4/_{11}$. So the limiting distribution is

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = \big(0, 0, {}^4/_{11}, {}^3/_{11}, {}^4/_{11}\big).$$

2 pt

(c) [3 pt.] Let m_i denote the expected number of steps it takes to reach state 3 starting from state i. Then, by conditioning on the first step of the Markov chain,

$$m_1 = 1 + \tfrac{2}{3}m_2 \qquad m_2 = 1 + \tfrac{1}{3}m_1$$

1 pt

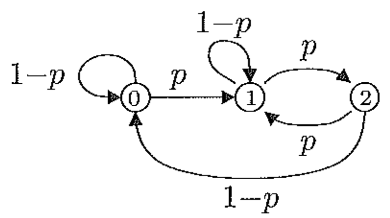
Substituting the second equation into the first gives $m_1 = \frac{5}{3} + \frac{2}{9}m_1$, hence

$$m_1 = {}^{15}/_7.$$

2 pt

2.

(a) [3 pt.] Let X_n = # of broken bikes waiting to be repaired at end of day n. This gives a DTMC on the state space $I = \{0, 1, 2\}$ with the following transition diagram (note that on the day that bikes are repaired, it is still possible that a broken bike is returned): } 1 pt



} 2 pt

(b) [4 pt.] The equilibrium equations are

$$\begin{aligned}
\pi_0 &= (1-p)\pi_0 + (1-p)\pi_2 &&\rightarrow p\pi_0 = (1-p)\pi_2 \\
\pi_1 &= p\pi_0 + (1-p)\pi_1 + p\pi_2 &&\rightarrow p\pi_1 = p\pi_0 + p\pi_2 \\
\pi_2 &= p\pi_1
\end{aligned}$$

} 2 pt

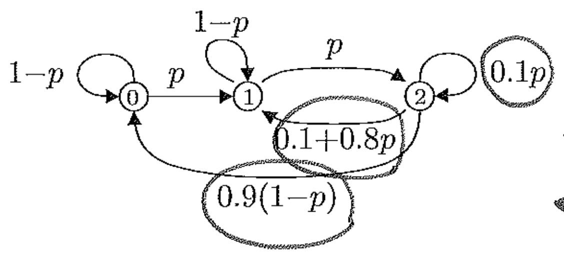
Combining the third and the first equation yields $\pi_0 = (1-p)\pi_1$, so by normalization,

$$\pi_0 + \pi_1 + \pi_2 = \pi_1((1-p) + 1 + p) = 2\pi_1 = 1.$$

} 1pt

Hence $\pi_1 = 1/2$ and π_2, which is the long-term fraction of days on which repairs are carried out, equals $p/2$. } 1 pt

(c) [3 pt.] The new situation can only change the transitions *from* state 2. We now return to state 0 only if the shop repairs both bikes, and no new broken bike is returned; if the shop manages to repair only one bike, and a new broken bike is brought in, we return to state 2. Since $p_{20} + p_{21} + p_{22}$ must be 1, this gives the following modified transition diagram:



← 1 pt

← 1 pt

← 1 pt

3.

(a) [3 pt.] By thinning, high- and low-priority jobs arrive according to independent Poisson processes $N_H(t)$ and $N_L(t)$, with respective rates of $\lambda/5$ and $4\lambda/5$ per millisecond. Hence, the required probability is 1pt

$$P(N_H(5) = 1, N_L(5) \le 2) = P(N_H(5) = 1)\, P(N_L(5) \le 2)$$ $\leftarrow$ 1pt

$$= \lambda e^{-\lambda} \times \left(1 + 4\lambda + \tfrac{1}{2} \cdot (4\lambda)^2\right) e^{-4\lambda}$$ $\leftarrow$ 1pt

(b) [4 pt.] Using lack of memory, the execution times of the job the processor is currently working on, and of the next job waiting in the queue must both (sequentially) "beat" the time we have to wait for the next job to arrive at the task scheduler. After that, the time until the next high-priority job arrives must "beat" both the execution time of the second job waiting in the queue, and the time until the next low-priority job arrives. 2pt

The probability of this happening is

$$\left(\frac{\mu}{\mu+\lambda}\right)^2 \times \frac{\lambda/5}{\mu + \lambda/5 + 4\lambda/5} = \frac{\lambda\mu^2}{5(\mu+\lambda)^3}$$ 2pt

4.

(a) [3 pt.] Let $X(t) = \#$ of production lines Up at time t. Then, following the description of the situation, $\{X(t), t \ge 0\}$ is a CTMC on the state space $I = \{0, 1, 2, 3\}$ with the following transition rate diagram: 1pt

$$\begin{tikzcd} 0 \arrow[r, bend left, "3\mu"] & 1 \arrow[l, bend left, "\lambda"] \arrow[r, bend left, "2\mu"] & 2 \arrow[l, bend left, "2\lambda"] \arrow[r, bend left, "\mu"] & 3 \arrow[l, bend left, "2\lambda"] \end{tikzcd}$$ 2pt

(b) [3 pt.] Let m_i denote the expected time it takes to reach state 1 starting from state i. By conditioning on the next state the CTMC will jump to, it then follows that

$$m_2 = \frac{1}{\mu+2\lambda} + \frac{\mu}{\mu+2\lambda} m_3$$

$$m_3 = \frac{1}{2\lambda} + m_2$$ 1pt

Substituting the first equation into the second gives

$$m_3 = \frac{1}{2\lambda} + \frac{1}{\mu+2\lambda} + \frac{\mu}{\mu+2\lambda} m_3$$ 2pt

hence

$$m_3 = \frac{\mu+2\lambda}{2\lambda}\left(\frac{1}{2\lambda} + \frac{1}{\mu+2\lambda}\right) = \frac{\mu+2\lambda}{4\lambda^2} + \frac{1}{2\lambda} = \frac{\mu+4\lambda}{4\lambda^2}$$

(c) [4 pt.] Since the Markov chain is irreducible and positive recurrent, it has a limiting distribution $\mathrm{p} = (p_0, p_1, p_2, p_3)$, and the long-run fraction of time the CMTC spends in state i is p_i. By the ergodic theorem, the long-run average costs per time unit (expressed in terms of the p_i) are 1 pt

$$3kp_0 + (c+2k)p_1 + (2c+k)p_2 + 2cp_3$$

The limiting distribution itself is determined by the balance equations 1 pt

$$\begin{aligned} 3\mu p_0 &= \lambda p_1 & (\mu+2\lambda)p_2 &= 2\mu p_1 + 2\lambda p_3 \\ (2\mu+\lambda)p_1 &= 3\mu p_0 + 2\lambda p_2 & 2\lambda p_3 &= \mu p_2 \end{aligned}$$

Note that the second equation on the left can be simplified using the first equation on the left, and that the same equation is obtained when you simplify the first equation on the right with the help of the second equation on the right. Hence the system of balance equations reduces to 1 pt

$$3\mu p_0 = \lambda p_1 \qquad 2\mu p_1 = 2\lambda p_2 \qquad 2\lambda p_3 = \mu p_2$$

This allows us to express p_1, p_2, p_3 in terms of p_0:

$$p_1 = \frac{3\mu}{\lambda}p_0 \qquad p_2 = \frac{\mu}{\lambda}p_1 = \frac{3\mu^2}{\lambda^2}p_0 \qquad p_3 = \frac{\mu}{2\lambda}p_2 = \frac{3\mu^3}{2\lambda^3}p_0$$

Normalization now gives

$$p_0 + p_1 + p_2 + p_3 = \frac{2\lambda^3 + 6\mu\lambda^2 + 6\mu^2\lambda + 3\mu^3}{2\lambda^3}p_0 = 1$$

hence

$$p_0 = \frac{2\lambda^3}{D} \qquad p_1 = \frac{6\mu\lambda^2}{D} \qquad p_2 = \frac{6\mu^2\lambda}{D} \qquad p_3 = \frac{3\mu^3}{D}$$

with the same denominator $D = 2\lambda^3 + 6\mu\lambda^2 + 6\mu^2\lambda + 3\mu^3$ in all cases. Thus the long-term average costs per time unit are 1 pt

$$6\,\frac{k\lambda^3 + (c+2k)\mu\lambda^2 + (2c+k)\mu^2\lambda + c\mu^3}{2\lambda^3 + 6\mu\lambda^2 + 6\mu^2\lambda + 3\mu^3}$$