# Resit Stochastic Modeling (400646)

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This exam consists of five exercises. The use of books or a graphical calculator is not allowed. A formula sheet can be found at the end of the exam.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for this exam is given by $p \times 9/33 + 1$ where $p$ is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

## Exercise 1

Consider a discrete-time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and matrix of transition probabilities

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) [2 pt.] Draw the state diagram of possible transitions and determine the classes of communicating states. What is $\mathbb{P}(X_3 = 5 \mid X_0 = 1)$?

(b) [3 pt.] Does the limiting distribution exist? And the occupancy distribution? If they exist, determine the limiting and/or the occupancy distribution.

(c) [2 pt.] Determine the expected number of transitions to reach state 5, starting in state 1.

## Exercise 2

At the railway station in a large city, it is possible to rent an OV rental bike at three different locations. The three locations are Central Station (CS), South (S), and West (W). When the OV rental bike is hired out at CS, it is returned at CS, S, and W with probabilities 0.7, 0.2, and 0.1, respectively. When the OV rental bike is hired out at

S, it is returned at CS or S with probabilities 0.3 and 0.7, respectively. When the OV rental bike is hired out at W, it returned at CS, S, and W with probabilities 0.3, 0.1, and 0.6, respectively. For simplicity, we assume that an OV rental bike is always rented out for exactly 1 day.

(a) [3 p.] Formulate an appropriate discrete-time Markov chain to model the location of an OV rental bike. Give the balance equations and determine the long-run fraction of times that an OV rental bike is returned at CS in terms of the limiting distribution.

(b) [3 p.] When an OV rental bike is returned at CS it is sometimes going in repair before it can be hired out again. At CS the probability of repair is $p$. Repair always takes exactly 2 days, after which it can be directly hired out again from CS. Extend the state space of the Markov chain in (a) to model the new situation. Specify the matrix of one-step transition probabilities.

## Exercise 3

A delivery service transports packets using trucks. These trucks arrive at random to pick up packets. A packet which is to be delivered from A to B can be transported directly, or via a distribution center (DC), whichever occurs first. Trucks going directly from A to B arrive according to a Poisson process with rate $\lambda$. Trucks going from A to DC arrive according to an independent Poisson process with rate 2. We are interested in the time until packets leave location A.

(a) [3 p.] What is the probability that after three time units the packet is still at A? And what is the probability that the packet is transported via the DC?

Suppose that the schedule is changed. Now, there is no direct transport from A to B, i.e. $\lambda = 0$. The interarrival time between two trucks from A to DC follows with probability $1/2$ an exponential distribution with rate 1, and with probability $1/2$ it follows an exponential distribution with rate 3.

(c) [3 pt.] Show that the probability that a packet at A arriving at an arbitrary moment has to wait *at least* $x$ time units before it is picked up is equal to

$$\frac{3}{4}e^{-x} + \frac{1}{4}e^{-3x}.$$

Also, intuitively explain the form of the above result.

## Exercise 4

The region of Amsterdam wants to invest in new ambulances that can transport intensive care patients. This ambulance is called a Mobile Intensive Care Unit (MICU). Management wants to know how many MICUs are required. Measurements show that requests for MICU transport arrive according to a Poisson process with an average of 16 patients per day. The total duration to transport a patient is approximated by an expotential distribution with an average of 2 hours. Requests for MICU transport when the MICU is occupied wait until the MICU becomes available again (in a FCFS manner).

(a) [4 p.] How many MICUs are required in order to ensure that the average waiting time is less than 3 hours?

Management decides to buy 1 MICU. However, new arriving requests for ambulance transport are rejected when the ambulance is occupied and there are 2 patients waiting for transport next to patient being transported (transport is then carried out by other means).

(b) [3 p.] Formulate a continuous-time Markov chain to model the number of patients for MICU transport. Draw the state diagram with corresponding transition rates and determine the fraction of arriving patients that are rejected.

## Exercise 5

Consider an M/G/1 queue with arrival rate $3/8$. With probability $p \in [0, 1]$ the service time follows an exponential distribution with rate $p$ and with probability $1 - p$ the service time is exactly $1/(1-p)$.

(a) [3 pt.] Show that the expected service time is 2. Derive the expected waiting time $\mathbb{E}[W^q]$ and verify that it is of the form

$$\mathbb{E}[W^q] = \frac{c_1 + c_2 p}{p(1-p)},$$

for some constants $c_1$ and $c_2$. Give those $c_1$ and $c_2$.

(b) [2 pt.] Determine the expected number of customers in the system. Also, explain the behavior of $\mathbb{E}[W^q]$ as $p \to 1$.

When the server is idle, it is always working on other tasks. The service time of these other tasks follows an exponential distribution with rate $\theta$. When a new customer arrives while the server is working on other tasks, the server completes the task and then returns to serve customers again.

(c) [2 pt.] Give the arrival relation for the expected waiting time of customers. Use this relation to derive its new expected waiting time.

**FORMULA SHEET**

**Erlang distribution** Let $S_n$ follow an Erlang($n$, $\mu$) distribution. The tail probability of $S_n$ is then

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!}.$$

**M/M/c queue** The probability of waiting $\Pi_W$, expectation and distribution of the waiting time $W^q$ and distribution of the sojourn time $S$

$$\begin{aligned}
\Pi_W &= \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!} \\
\mathbb{E}(W^q) &= \Pi_W \frac{1}{c\mu(1-\rho)} \\
\mathbb{P}(W^q > t) &= \Pi_W e^{-c\mu(1-\rho)t} \\
\mathbb{P}(S > t) &= \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}
\end{aligned}$$

**M/G/c/c queue** Blocking probability $B(c,a)$, with $a = \lambda \mathbb{E}B = c\rho$,

$$B(c,a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!}$$

**M/G/1 queue** Expected waiting time $W^q$ for FCFS (Pollaczek-Khintchine)

$$\mathbb{E}(W^q) = \frac{\rho}{1-\rho} \frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)} = \frac{1}{2}(1 + c_B^2)\mathbb{E}(B)\frac{\rho}{1-\rho}$$

Expected busy period

$$\mathbb{E}(BP) = \frac{\mathbb{E}(B)}{1-\rho}$$

**G/M/1 queue** Distribution number of customers found upon arrival $\pi^*$ and expected waiting time $W^q$

$$\pi_j^* = (1-\sigma)\sigma^j \qquad \text{and} \qquad \mathbb{E}(W^q) = \frac{\sigma}{\mu(1-\sigma)}$$

with $\sigma$ unique solution in $(0,1)$ of $\sigma = \mathbb{E}\left[e^{-\mu(1-\sigma)A}\right]$ with $A$ interarrival time

**Residual life time** Let $X$ be the interarrival time and $R$ be the residual life time. Distribution and expectation of the residual life time $R$

$$\mathbb{P}(R \leq x) = \frac{1}{\mathbb{E}(X)} \int_0^x \mathbb{P}(X > y)\mathrm{d}y \qquad \text{and} \qquad \mathbb{E}(R) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}$$