Exam Stochastic Modeling (400646), period 2

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December 22, 2017, 8:45 - 10:45 hours

This exam consists of three exercises. The use of books or a graphical calculator is not allowed. A formula sheet can be found at the end of the exam.

At each part it is indicated between square brackets how many points can be achieved for the corresponding part. The grade for this exam is given by $p/3 + 1$ where p is the total number of points earned by you. Please include your name and student number on all papers and motivate all your answers clearly.

Good luck!

Exercise 1

Consider the following M/M/2/K queue. Customers arrive according to a Poisson process with rate λ. The service times of customers follow an exponential distribution with rate μ. There are 2 servers and there is place for K customers in the system in total. Arriving customers that find K customers upon arrival are lost.

(a) [4 pt.] For which values of λ and μ is the system stable? Let p_i be the limiting distribution of having i customers in the system. Specify the state diagram with the transition rates and express p_i in terms of p_0.

(b) [2 pt.] Suppose that $K = 4$. Determine p_0 and the fraction of customers that is lost in terms of λ and μ.

(c) [2 pt.] Suppose again that $K = 4$. Determine the probability that a customer has to wait at least t time units before receiving service.

Exercise 2

A company at Schiphol airport rents cars. The company has c cars. Customers arrive according to a Poisson process with a rate of 4 customers per day. A customer rents a car for an exponential time with a mean of 1.5 days. Customers that arrive when there are no cars available are lost and go to another company.

(a) [3 p.] Formulate a continuous-time Markov chain to analyze the number of cars available and give the corresponding state diagram with the transition rates. Give an expression for the fraction of customers for which no car is available.

(b) [2 p.] A customer is interested in a specific car that is currently being rented, and this specific customer decides to wait until this car is available. What is the probability that the customer has to wait at most 3 days before the customer can pick up the car? Clearly motivate your answer.

Due to marketing, the company is able to attract a second group of customers. These customers arrive according to a Poisson process with a rate of 2 per day and they rent a car for a period of time that is uniformly distributed between 1 and 5 days.

(c) [3 p.] Explain that the new situation can be modeled as an M/G/c/c queue. Give an expression for the fraction of customers for which no car is available for this new situation.

Exercise 3

Consider a queueing system with a single server and infinite waiting capacity. Customers arrive according to a Poisson process with rate $\frac{2}{3}$. Customers may require three different types of services. With probability $p \in (0, 0.5]$ service is of type 1 and takes *exactly* 0.5 time units. With probability p service is of type 2 and follows an exponential distribution with a mean of 1 time unit. With probability $1-2p$ service is of type 3 and takes *exactly* 1.5 time units. Customers are served First-Come-First-Served (FCFS).

(a) [4 pt.] Show that the expected service time ($\mathbb{E}B$) is $\frac{3}{2}(1-p)$ and that the expected waiting time $\mathbb{E}[W^q]$ is given by

$$\mathbb{E}[W^q] = \frac{3}{4}\frac{1-p}{p}.$$

(b) [3 pt.] Make a sketch of $\mathbb{E}[W^q]$ as a function of p, for $p \in (0, 0.5]$, and explain its behavior. Also determine the expected number of customers in the queue, the expected sojourn time, and the expected number of customers in the system.

The service discipline is changed into Last-Come-First-Served Preemptive Resume (LCFS-PR).

(c) [3 pt.] Consider a customer of type 2 and with a required service time of x. Argue that the arrival relation for the expected sojourn time ($\mathbb{E}S(x)$) is

$$\mathbb{E}S(x) = x + \frac{2}{3}x\,\mathbb{E}BP,$$

with BP the busy period of this M/G/1 queue. Show that the expected sojourn time for a type-2 customer with a required service time of x is equal to x/p. Give the expected sojourn time of an arbitrary type-2 customer.

(d) [1 pt.] Give the arrival relation for the sojourn time of a type 1 customer and determine its expected sojourn time.

FORMULA SHEET

Erlang distribution Let S_n follow an Erlang(n, μ) distribution. The tail probability of S_n is then

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1} e^{-\mu t} \frac{(\mu t)^k}{k!}.$$

M/M/c queue The probability of waiting Π_W, expectation and distribution of the waiting time W^q and distribution of the sojourn time S

$$\begin{aligned}
\Pi_W &= \frac{(c\rho)^c/c!}{(1-\rho)\sum_{i=0}^{c-1}(c\rho)^i/i! + (c\rho)^c/c!} \\
\mathbb{E}(W^q) &= \Pi_W \frac{1}{c\mu(1-\rho)} \\
\mathbb{P}(W^q > t) &= \Pi_W e^{-c\mu(1-\rho)t} \\
\mathbb{P}(S > t) &= \frac{\Pi_W}{1-c(1-\rho)} e^{-c\mu(1-\rho)t} + \left(1 - \frac{\Pi_W}{1-c(1-\rho)}\right) e^{-\mu t}
\end{aligned}$$

M/G/c/c queue Blocking probability $B(c,a)$, with $a = \lambda \mathbb{E}B = c\rho$,

$$B(c,a) = \frac{a^c/c!}{\sum_{i=0}^{c} a^i/i!}$$

M/G/1 queue Expected waiting time W^q for FCFS (Pollaczek-Khintchine)

$$\mathbb{E}(W^q) = \frac{\rho}{1-\rho}\frac{\mathbb{E}(B^2)}{2\mathbb{E}(B)} = \frac{1}{2}(1+c_B^2)\mathbb{E}(B)\frac{\rho}{1-\rho}$$

Expected busy period

$$\mathbb{E}(BP) = \frac{\mathbb{E}(B)}{1-\rho}$$

G/M/1 queue Distribution number of customers found upon arrival π^* and expected waiting time W^q

$$\pi_j^* = (1-\sigma)\sigma^j \qquad \text{and} \qquad \mathbb{E}(W^q) = \frac{\sigma}{\mu(1-\sigma)}$$

with σ unique solution in $(0,1)$ of $\sigma = \mathbb{E}\left[e^{-\mu(1-\sigma)A}\right]$ with A interarrival time

Residual life time Let X be the interarrival time and R be the residual life time. Distribution and expectation of the residual life time R

$$\mathbb{P}(R \le x) = \frac{1}{\mathbb{E}(X)} \int_0^x \mathbb{P}(X > y)\mathrm{d}y \qquad \text{and} \qquad \mathbb{E}(R) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}$$