Exam Stochastic Modeling (400646), period 2 - Solutions

The solutions are always provisionary

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Exercise 1.

(a) The system is stable for all λ and μ due to the finite state space. The state diagram with the transition rates is given in Figure 1.

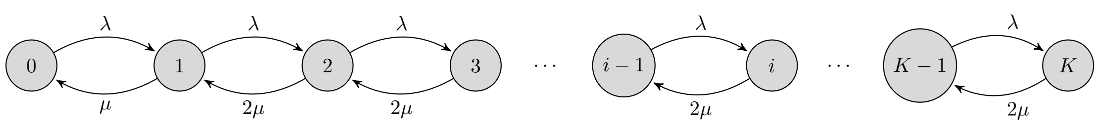

Figure 1: State diagram Exercise 1(a).

The balance equations are then $\lambda p_0 = \mu p_1$ and $\lambda p_{i-1} = 2\mu p_i$, for $i = 2, 3, \ldots, K$. From the first equation we directly obtain that $p_1 = {}^{\lambda}\!/\!_{\mu}\, p_0$. Moreover, for $i = 2, \ldots, K$,

$$p_i = \frac{\lambda}{2\mu} p_{i-1} = \left(\frac{\lambda}{2\mu}\right)^{i-1} p_1 = 2\left(\frac{\lambda}{2\mu}\right)^{i} p_0.$$

Note that the final result is also valid for $i = 1$, as the first step is then omitted. Hence, we now expressed all p_i in terms of p_0.

(b) We need to determine p_0 using normalization:

$$p_0 \Big(1 + \underbrace{\sum_{i=1}^{4} 2\left(\frac{\lambda}{2\mu}\right)^{i}}_{=\frac{\lambda}{\mu}\frac{1-(\lambda/2\mu)^4}{1-\lambda/2\mu}} \Big) = 1,$$

where there are different ways to (re)write the finite sum. After some calculus, it holds that

$$p_0 = \frac{1 - \lambda/2\mu}{1 + \lambda/2\mu - 2(\lambda/2\mu)^5}.$$

Due to PASTA, the fraction of customers lost is

$$p_4 = 2\left(\frac{\lambda}{2\mu}\right)^{4} \frac{1 - \lambda/2\mu}{1 + \lambda/2\mu - 2(\lambda/2\mu)^5}.$$

(c) The key element is to condition on the number of customers found upon arrival. If, upon arrival, there are 2 customers, the waiting time is Exp(2μ); if there are 3 customers upon arrival, the waiting time is Erlang(2, 2μ). If there are 0, 1 or 4 customers, there is no waiting time. Thus,

$$\mathbb{P}(W^q > t) = p_2 e^{-2\mu t} + p_3 e^{-2\mu t}(1 + 2\mu t),$$

where p_2 and p_3 are given in parts a and b.

Exercise 2.

(a) Let $X(t)$ be the number of cars on rent at time t. Then $\{X(t), t \geq 0\}$ is a CTMC on state space $\{0, 1, \ldots, c\}$, with the tranistion diagram given in Figure 2.

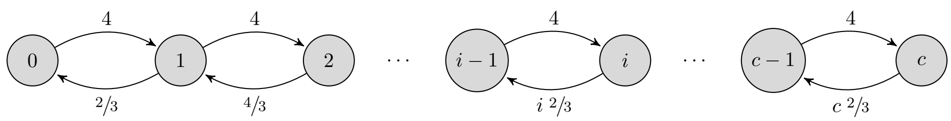

Figure 2: State diagram Exercise 2(a).

Observe that the model corresponds to the Erlang B model, where the offered load $a = 4 \times {}^3\!/_2 = 6$. Hence, the fraction of customers for which no car is availble is

$$B(c, 6) = \frac{6^c/(c!)}{\sum_{i=0}^{c} 6^i/(i!)}.$$

(b) Observe that the time until the car is available corresponds to a residual service time R. For Exp(${}^2\!/_3$) service times, it holds that $R \sim$Exp(${}^2\!/_3$). Thus, the required waiting time is $\mathbb{P}(R \leq 3) = 1 - e^{-3\times {}^2\!/_3} = 1 - e^{-2}$.

(c) The system still corresponds to a multi-server system (where the cars are servers) with no waiting line. The arrival process is the superposition of two Poisson processes, which is again a Poisson process. The service time is now a mixture of Exp(${}^2\!/_3$), with probability ${}^2\!/_3$ and a Unif(1, 5), with probability ${}^1\!/_3$. This mixture can be considered as a general distribution.

We thus only need the new total offered load, which is $a = 6 + 2 \times 3 = 12$. The fraction of customers for which no car is available is then

$$B(c, 12) = \frac{12^c/(c!)}{\sum_{i=0}^{c} 12^i/(i!)}.$$

Exercise 3.

(a) The expected service time is obtained by conditioning on the type of service:

$$\mathbb{E}B = p \times \frac{1}{2} + p \times 1 + (1 - 2p) \times \frac{3}{2} = \frac{3}{2}(1 - p).$$

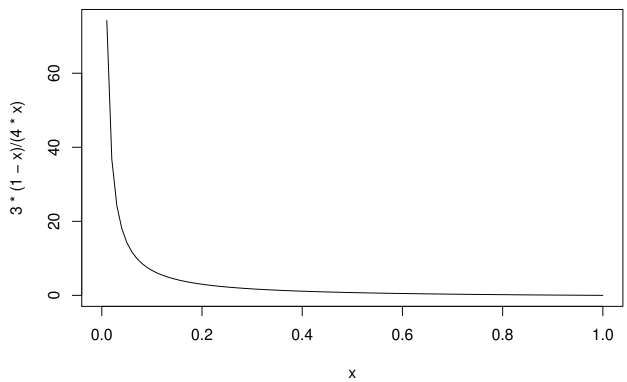

Figure 3: For exercise 3b, sketch of $\mathbb{E}W^q$ as a function of p.

Similarly, the second moment is

$$\mathbb{E}B^2 = p \times \left(\frac{1}{2}\right)^2 + p \times \frac{2}{1^2} + (1-2p) \times \left(\frac{3}{2}\right)^2 = \frac{9}{4}(1-p).$$

The load is $\rho = {}^2\!/\!_3 \times {}^3\!/\!_2(1-p) = 1-p$. Applying the Pollaczek-Khinchine formula and using the above gives

$$\mathbb{E}W^q = \frac{\rho}{1-\rho}\frac{\mathbb{E}B^2}{2\mathbb{E}B} = \frac{1-p}{p}\frac{{}^9\!/\!_4(1-p)}{2({}^3\!/\!_2(1-p))} = \frac{3}{4}\frac{1-p}{p}.$$

(b) See Figure 3 for a sketch of $\mathbb{E}W^q$ as a function of p. Note that the load is $\rho = 1-p$, or $p = 1-\rho$. We thus obtain the 'mirrored figure' of the classical sketch of $\mathbb{E}W^q$ for an M/G/1 queue as a function of the load ρ. This implies that $\mathbb{E}W^q$ is decreasing in p and $\mathbb{E}W^q$ tends to infinity for $p \downarrow 0$, as the system reaches its stability region. Moreover,

$$\begin{aligned}
\mathbb{E}L^q &= \lambda \mathbb{E}W^q = \frac{1}{2}\frac{1-p}{p} \\
\mathbb{E}S &= \mathbb{E}W^q + \mathbb{E}B = \frac{3}{4}\frac{1-p}{p} + \frac{3}{2}(1-p) \\
\mathbb{E}L &= \lambda \mathbb{E}S = \frac{1}{2}\frac{1-p}{p} + 1 - p.
\end{aligned}$$

(c) The elements in the arrival relation are explained as follows: x represents the customers own service time; ${}^2\!/\!_3\, x$ is the expected number of arrivals during such a service time; $\mathbb{E}BP$ corresponds to an 'extended' service time, i.e., the time required to serve an arriving customer and all customers that arrive before that particular leaves (thus a busy period).

Observe that $\mathbb{E}BP = \frac{\mathbb{E}B}{1-\rho} = {}^3\!/\!_2(1-p)/p$. Hence, the sojourn time of a customer of size x is

$$\mathbb{E}S(x) = x + \frac{2}{3}x\,\mathbb{E}BP = x + \frac{2}{3}x\,\frac{3}{2}\frac{1-p}{p} = x\left(1 + \frac{1-p}{p}\right) = \frac{x}{p}.$$

For the unconditional sojourn time, we obtain $\mathbb{E}S = \int_0^\infty {}^{x}\!/_{p}\, e^{-x} \mathrm{d}x = {}^{1}\!/_{p}$.

(d) The arrival relation for type 1 corresponds to part c with $x = {}^{1}\!/_{2}$:

$$\mathbb{E}S = \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \mathbb{E}BP.$$

Using the same expression for $\mathbb{E}BP$ as in part c, we have $\mathbb{E}S = {}^{1}\!/_{(2p)}$.